SCET-II: Endpoint Singularities & the Zero-Bin

SCET Workshop, LBL 2007

Manohar & I.S. hep-ph/0605001

Lange, Manohar, & I.S. in progress

Goal

Study factorization in cases with singular convolutions

$$\int_0^1 dx\, C(x)\phi_\pi(x) = \int_0^1 dx\, \frac{\phi_\pi(x)}{x^2} \sim \int_0 dx\, \frac{1}{x} = ?$$

√ • Define $\mathrm{SCET}_{\mathrm{II}}$ independent of UV and IR regulators (in particular should not rely on dim.reg.)

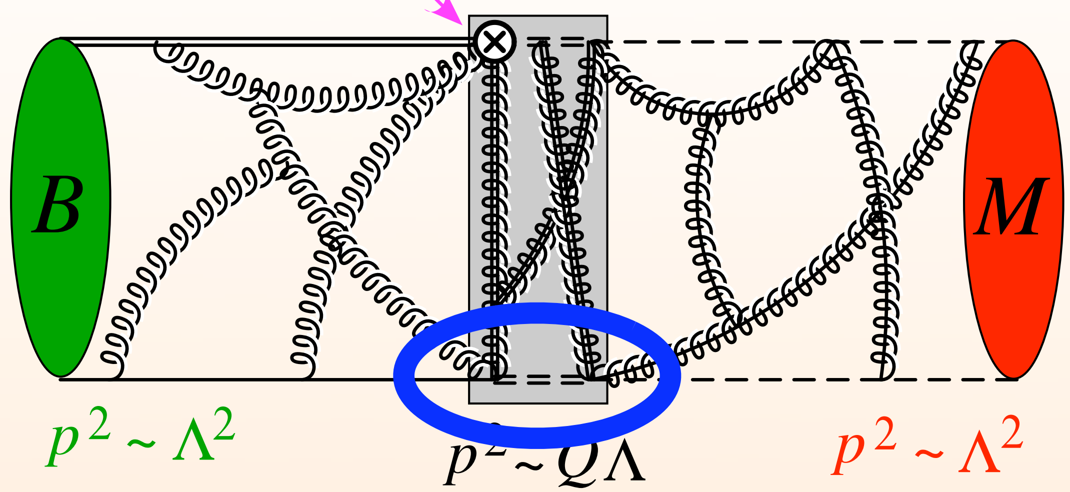
√ • Obtain finite EFT amplitudes & resolve singularity problem

• Derive factorization theorem that separates modes by both invariant mass and rapidity, with RG evolution etc.

$B \to \pi \ell \bar{\nu}$

Step 1:

$Q^2 \gg Q\Lambda$



Requires a power suppressed interaction

SCETI

needs time-ordered products

$$Q^{(0)} = \bar{\chi}_{n,\omega} \Gamma \mathcal{H}_v^n$$

$$Q^{(1)} = \bar{\chi}_{n,\omega} ig \mathcal{B}^{\perp}_{n,\omega'} \Gamma \mathcal{H}_v^n$$

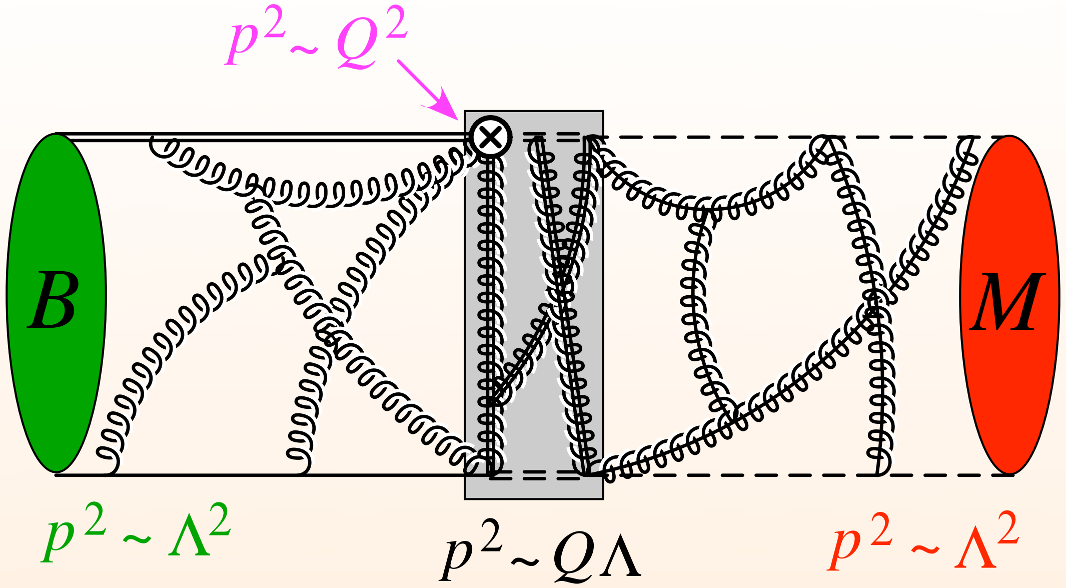
with

$$\mathcal{L}^{(1)}_{\xi q} = (\bar{q} Y) ig \mathcal{B}^{\perp}_{n,\omega'} \chi_n \,,$$

...

$$f(E) = \int dz\, T(z,E)\, \zeta_J^{BM}(z,E) \quad + C(E)\, \zeta^{BM}(E)$$

no singularity problem here

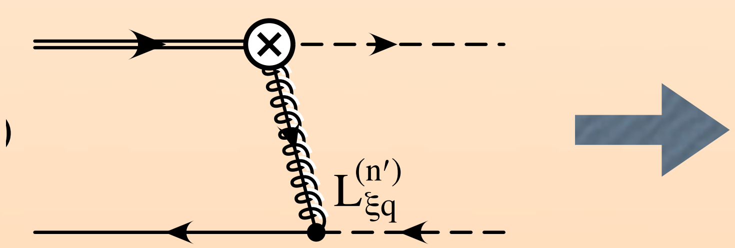


$$f(E) = \int dz\, T(z,E)\, \zeta_J^{BM}(z,E) + C(E)\, \zeta^{BM}(E)$$

Step 2: (further factorization) $Q\Lambda \gg \Lambda^2$ SCETII

ok: $$\zeta_J^{BM}(z) = f_M f_B \int_0^1 dx \int_0^\infty dk^+ J(z,x,k^+,E)\phi_M(x)\phi_B(k^+)$$

but: $\zeta^{BM} = ?$



$$\int_0^1 \mathrm{d}x\, \frac{\phi_\pi(x)}{x^2} = ???$$

endpoint singularity

one x from the Wilson line
one x from the gluon propagator

for phenomenology
$\zeta^{BM}(E)$ is left
unfactorized

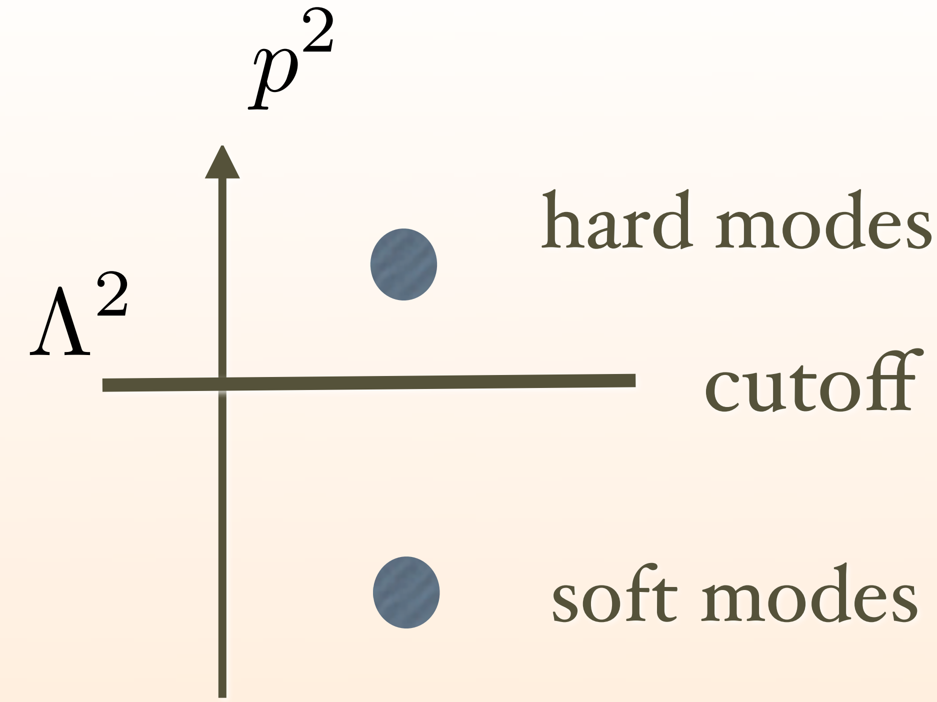
Wilsonian vs. Continuum EFT

Wilson

effective action for soft modes e^{-S_Λ}

removing modes with $\Lambda - \delta\Lambda < E < \Lambda$

$$e^{-S_{\Lambda-\delta\Lambda}} = \int_{\delta\Lambda} d\phi \; e^{-S_\Lambda}$$

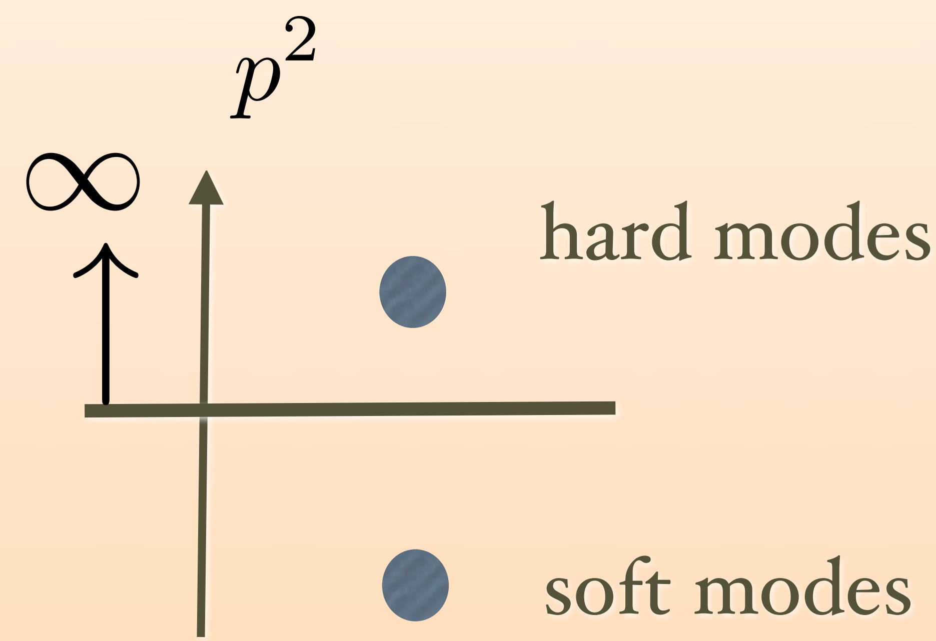


Continuum

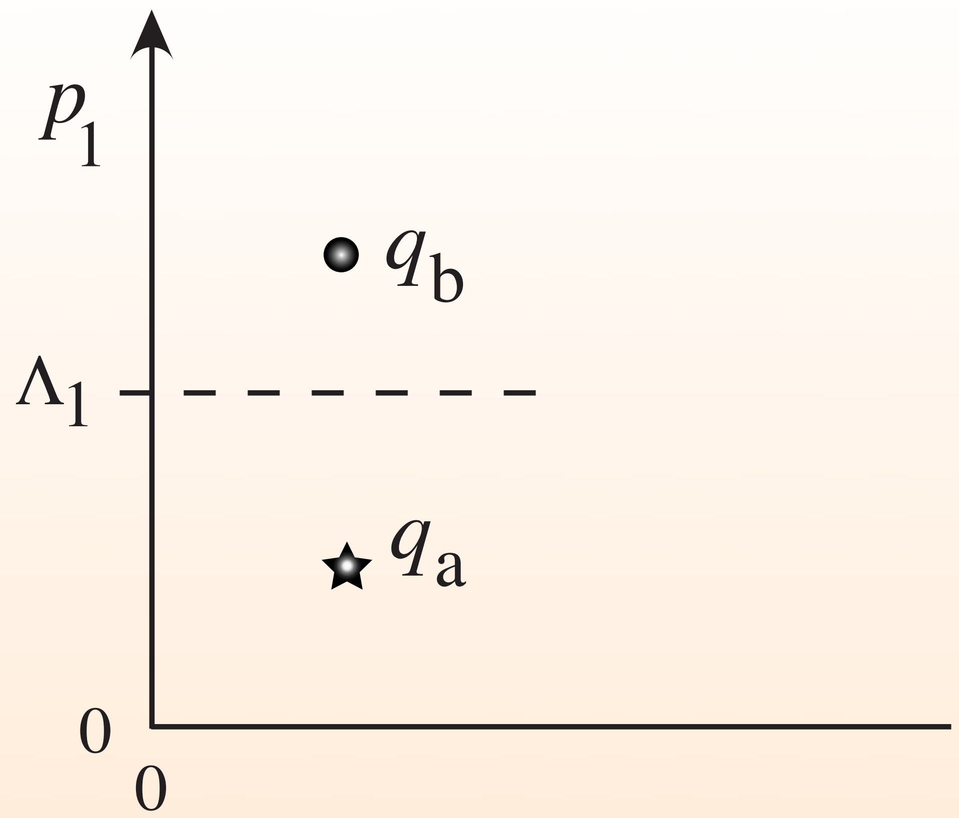
$$\mathcal{L}^{\mathrm{EFT}} = C(\mu)\mathcal{O}(\mu)$$

operators for soft modes $\mathcal{O}(\mu)$

Wilson coefficients for hard modes $C(\mu)$



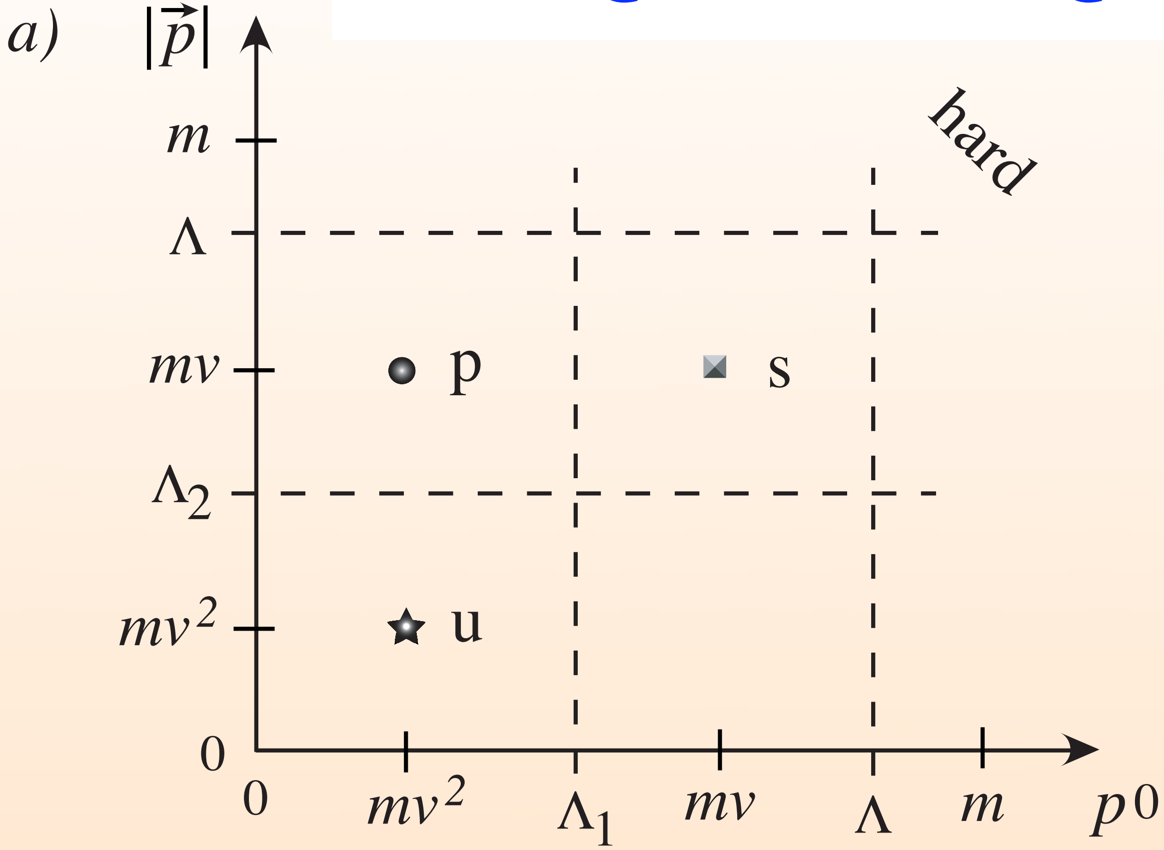
Sending Λ to ∞ includes the hard region in the matrix elements of our operators, but we fix $C(\mu)$ to correct for this.



Some EFT's need another dimension.

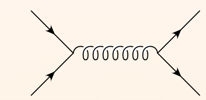
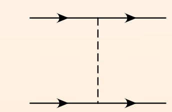
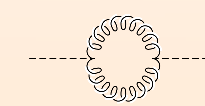
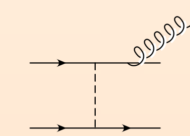
I'll call these "differential EFT's" .

NRQCD, NRQED

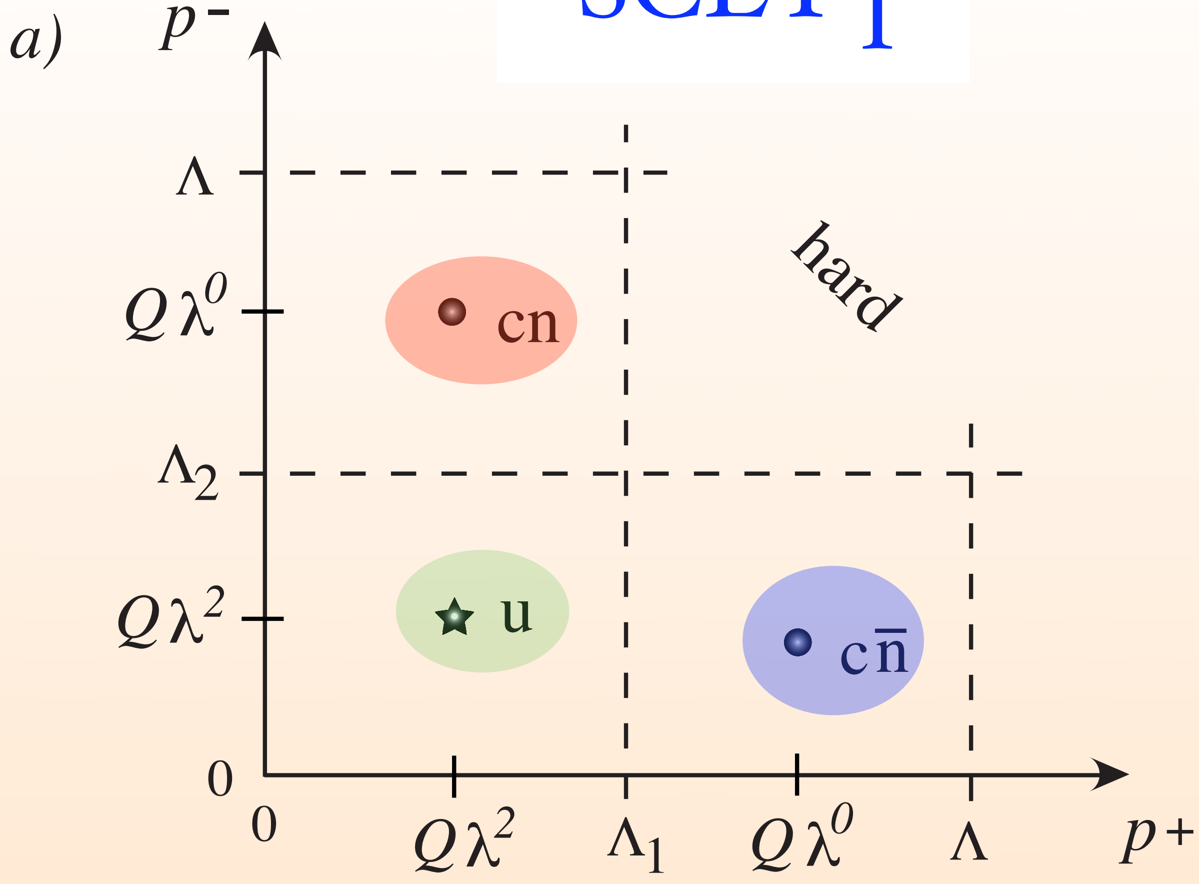


Non-relativistic field theory

Momentum Regions

	k^0	**k**	
hard:	m	m	
potential:	mv^2	mv	
soft:	mv	mv	
ultrasoft:	mv^2	mv^2	

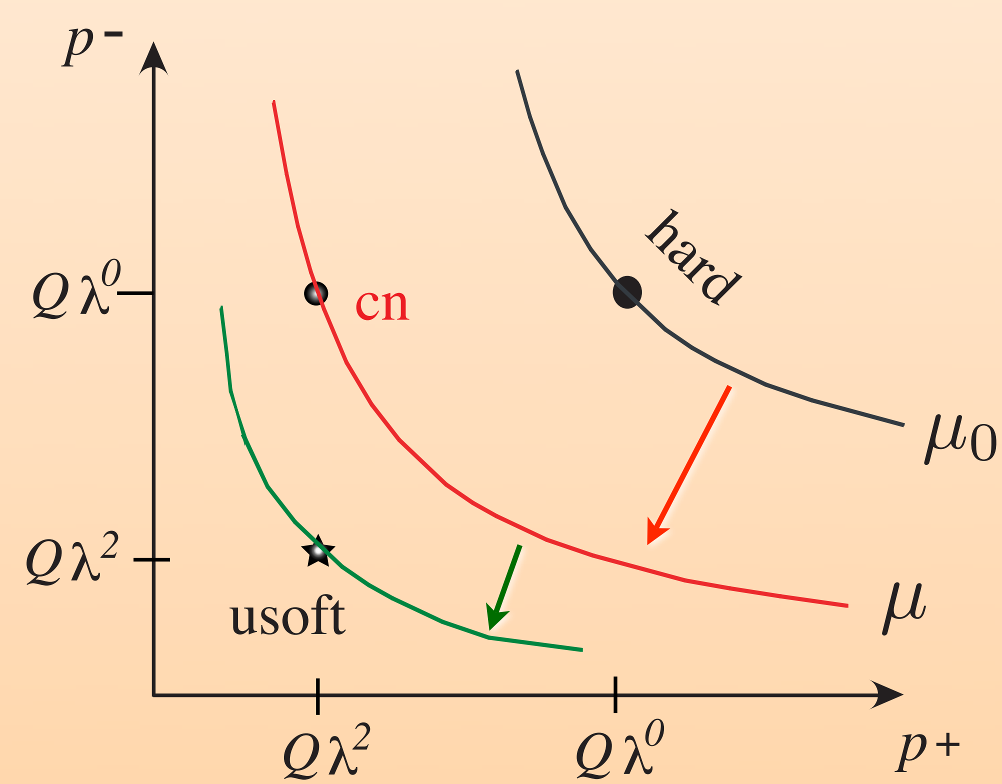
$\mathrm{SCET}_{\mathrm{I}}$



A formalism for jets.

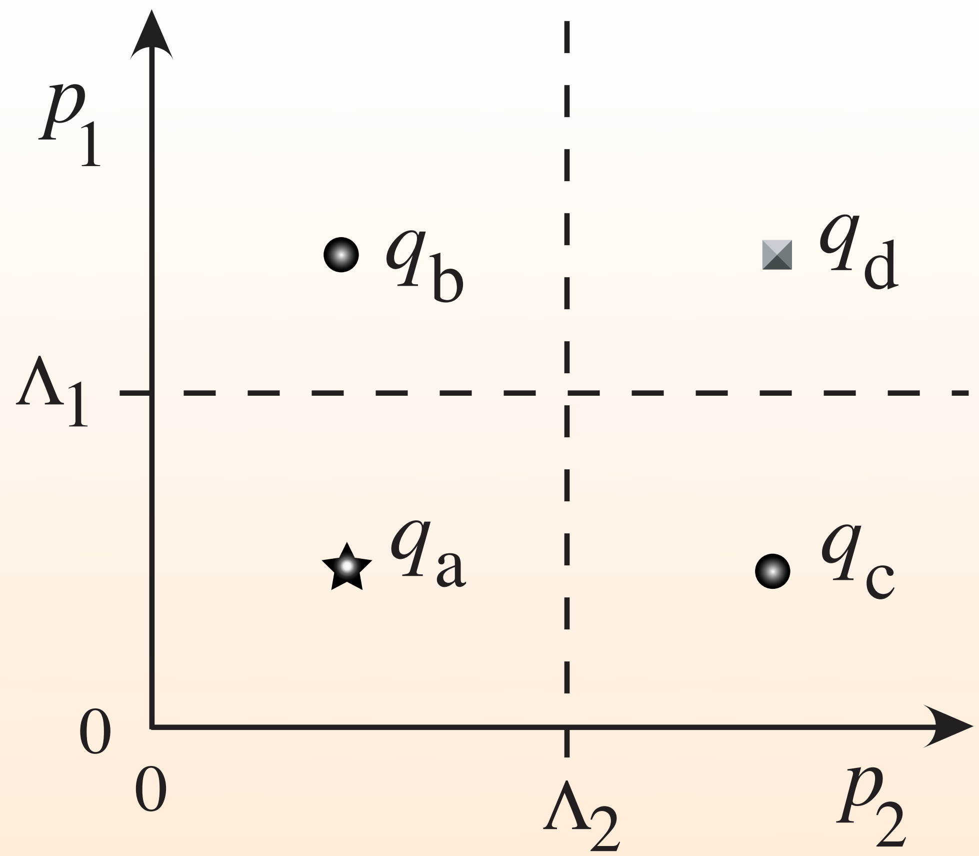
$$B \to X_s \gamma$$

$$e^+ e^- \to \text{two jets}$$

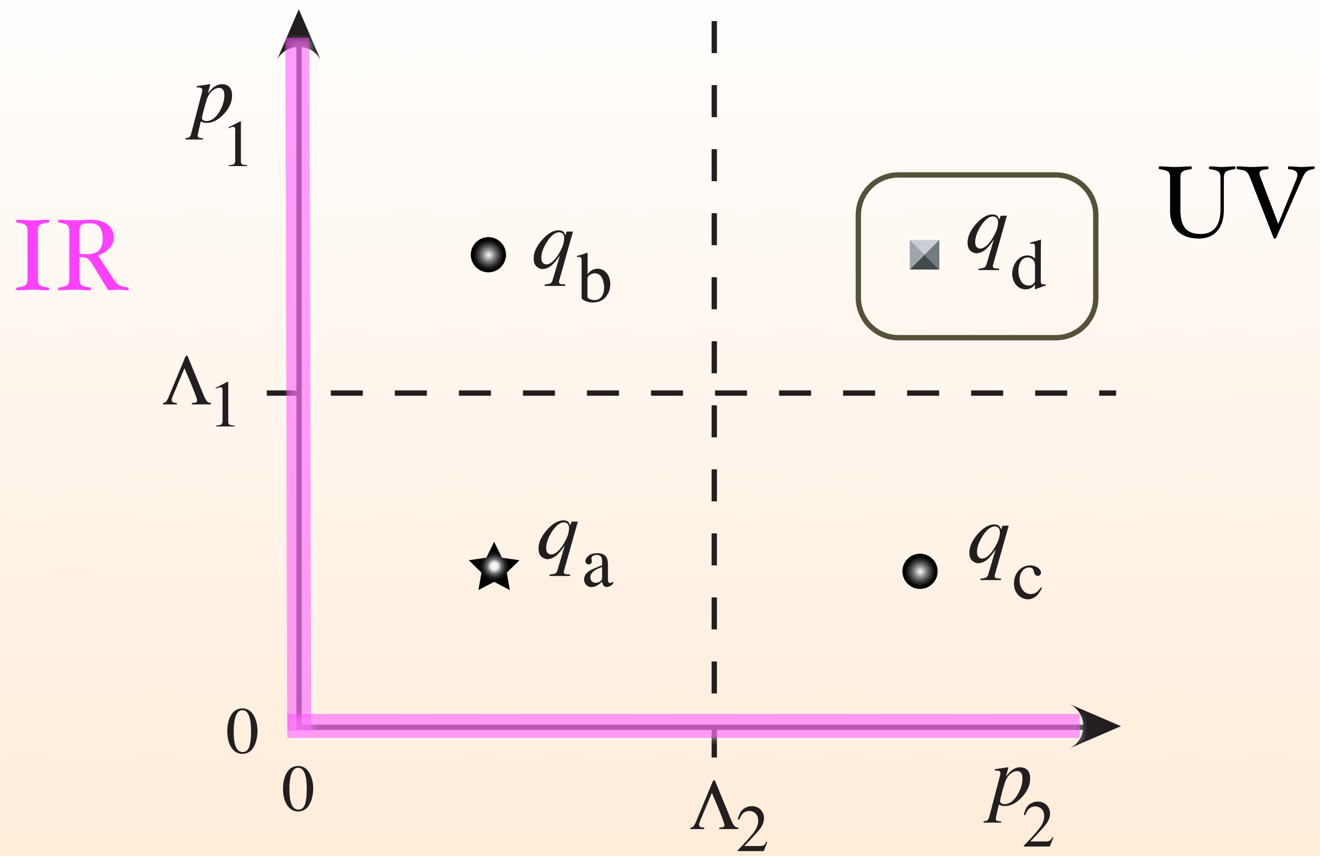


$$p^2 = p^+ p^- + p_\perp^2$$

$$m_X^2 \sim Q^2 \lambda^2 \gg \Lambda_{\mathrm{QCD}}$$

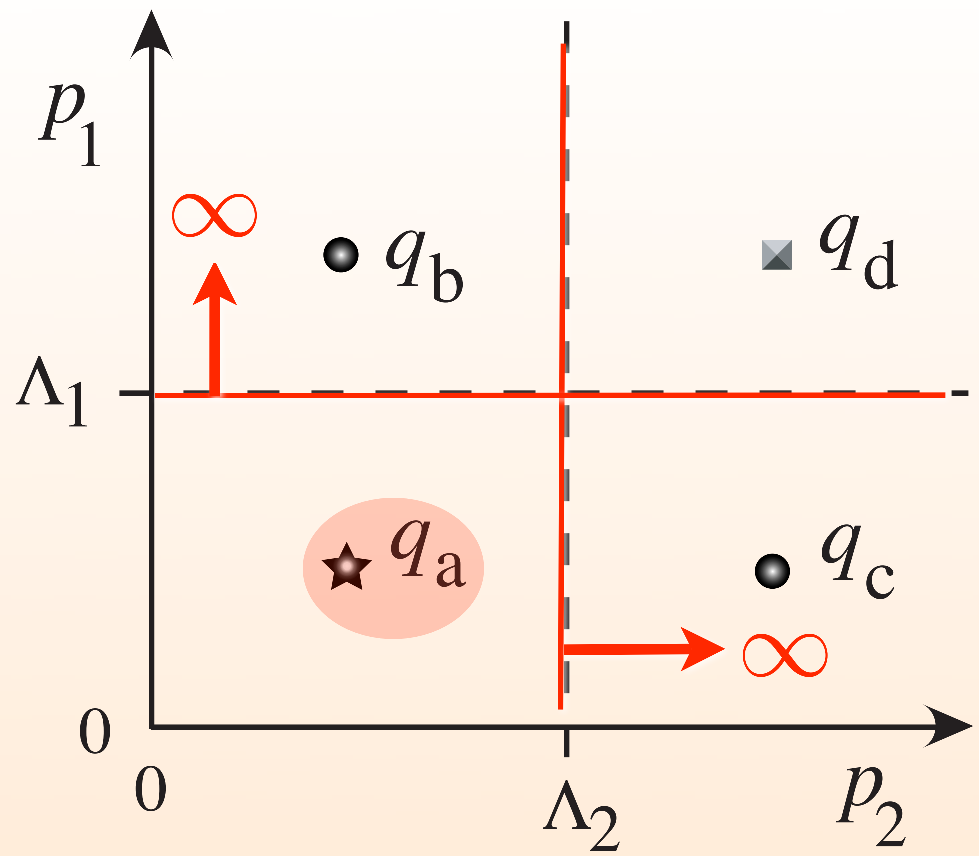


p_1
q_b
q_d
Λ_1
q_a
q_c
0
0
Λ_2
p_2



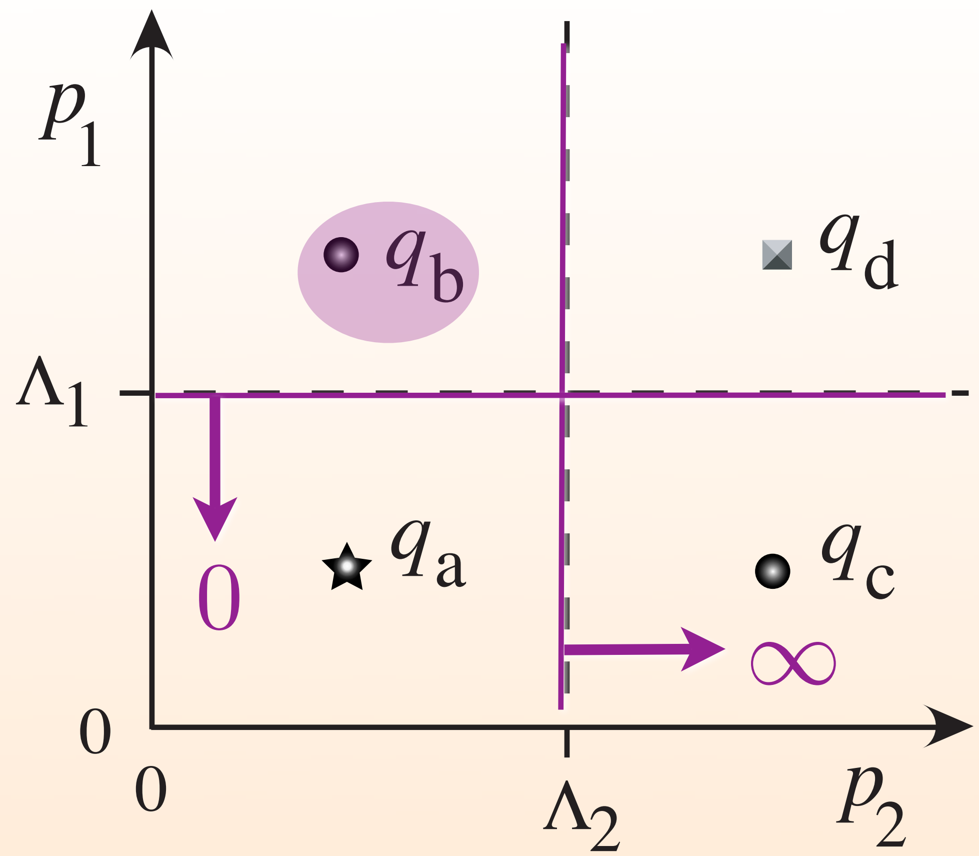
p_1
IR
UV
q_b
q_d
Λ_1
q_a
q_c
0
0
Λ_2
p_2

remove



- q_a overlaps only in the UV, fixed by Wilson coefficients

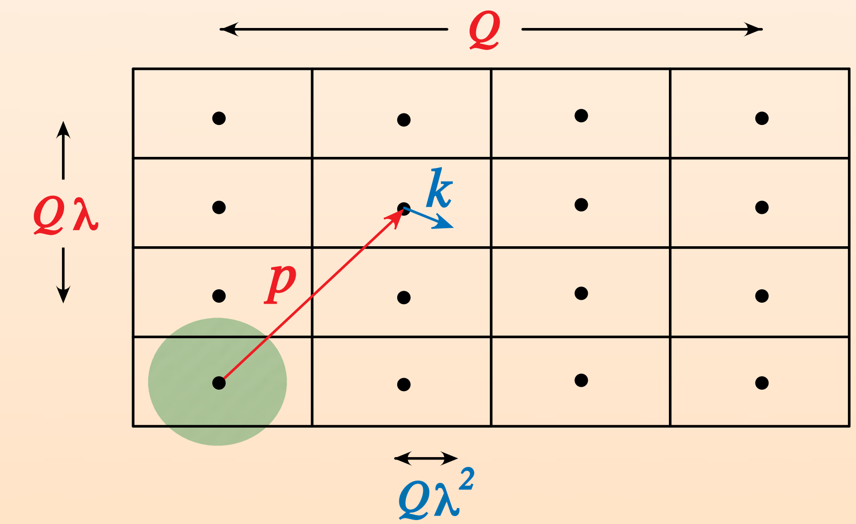
remove



label & residual momenta

$$P^\mu = p^\mu + k^\mu$$

$$\psi(x) \to \sum_{p \neq 0} e^{-ip\cdot x} \xi_{n,p}(x)$$



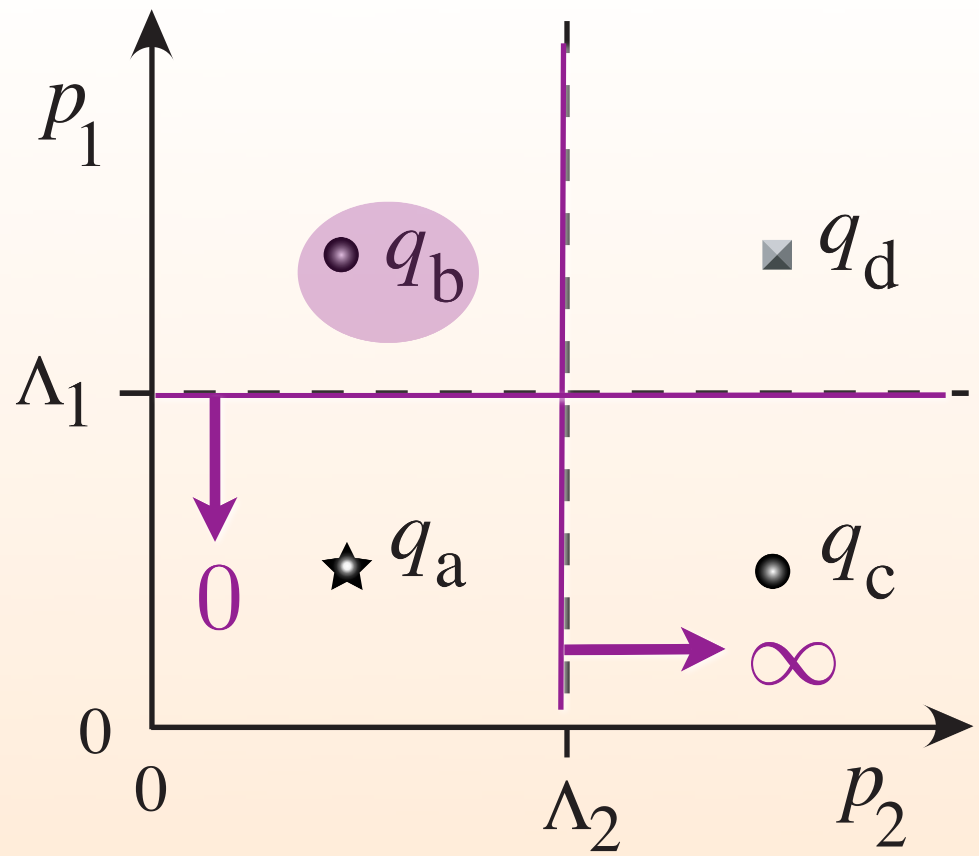
- q_b has label momentum $p_1 \neq 0$

$$\sum_{p_1 \neq 0} \int dp_{1r}\, F^{(q_b)}(p_1) = \int dp_1 \left[F^{(q_b)}(p_1) - F^{(q_b \to q_a)}_{\text{subt}}(p_1) \right]$$

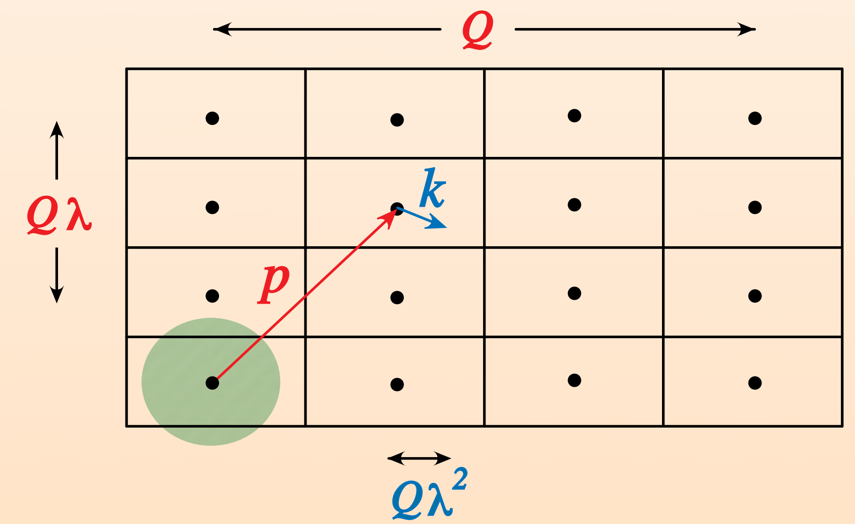
tiling formula

- symmetric story for q_c which has label momentum $p_2 \neq 0$

remove



- q_b has label momentum $p_1 \neq 0$



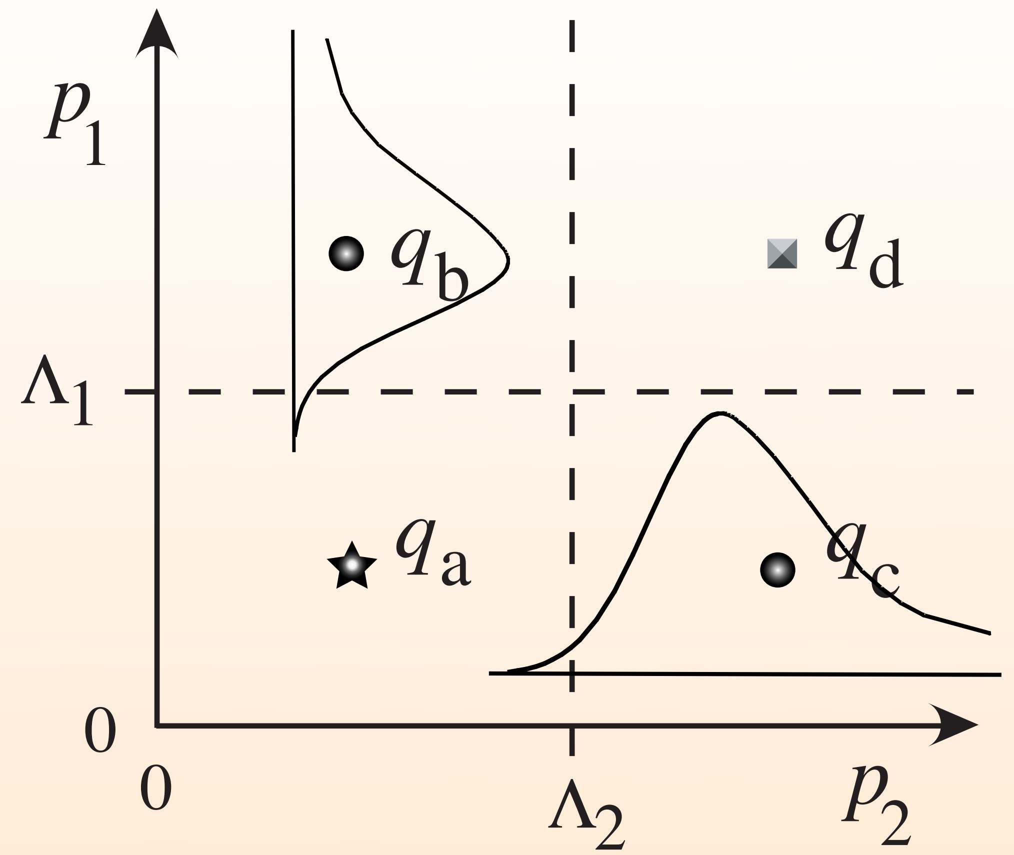
$$\sum_{p_1 \neq 0} \int dp_{1r}\, F^{(q_b)}(p_1) = \int dp_1 \left[F^{(q_b)}(p_1) - F^{(q_b \to q_a)}_{\rm subt}(p_1) \right]$$

tiling formula

Zero-bin subtractions

defined in collinear Lagrangian

For cases with singularities
the subtractions are needed
to not double count a region



Beyond this, different ways of implementing the subtractions correspond to a scheme dependence in defining the modes.

eg. Gaussians, hard cutoffs, ...

Lets define a nice, almost "scaleless", scheme like $\overline{\text{MS}}$:

- Take the integrand $F^{(q^b)}(p_1)$ constructed with the p.c. for its region.
- Expand this integrand with p_1 scaling as in region q_a and define $F_{subt}^{(q_b \to q_a)}$ by the terms up to marginal order in the power counting.

$$\sum_{p_1 \neq 0} \int dp_{1r}\, F^{(q_b)}(p_1) = \int dp_1 \Big[F^{(q_b)}(p_1) - F_{\text{subt}}^{(q_b \to q_a)}(p_1) \Big]$$

tiling formula

What has been done in the past?

$$\sum_p \int d^4k \longrightarrow \int d^dp$$ Ok if p=0 is harmless.

In cases where it is not harmless we exploited dimensional regularization:

Method of Regions (Beneke & Smirnov)

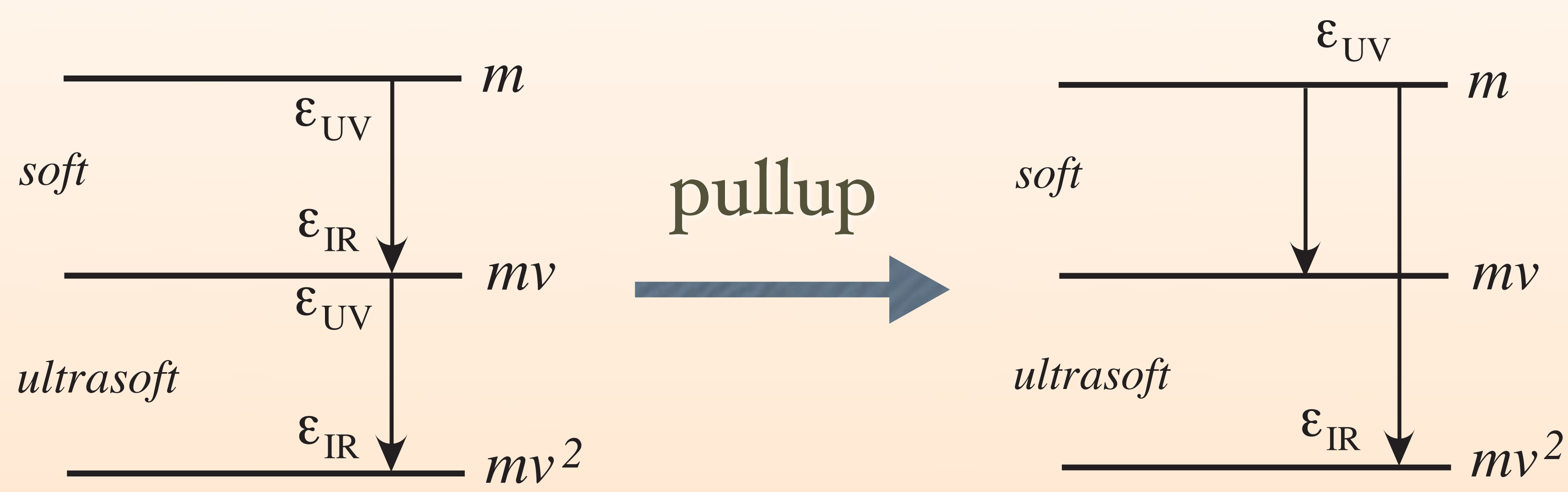
Any full theory loop integral depending on scales p_i satisfies:

$$\prod_j \int d^dk_j\, F(p_i, k_j) = \sum_{\text{regions } \ell} \prod_j \int d^dk_j\, F^{(\ell)}(p_i, k_j)$$

as long as we set $\epsilon_{\mathrm{IR}} = \epsilon_{UV} = \epsilon$ for every region

- Using this, the only errors one makes in defining the EFT modes are proportional to $\left(\frac{1}{\epsilon_{\rm UV}} - \frac{1}{\epsilon_{\rm IR}}\right)$. These can be fixed by hand, "a pullup", so that there is only one meaning for $\epsilon_{\rm UV}$.

Hoang, Manohar, I.S.



Not elegant, but it works.

- However, dim.reg. does not handle all singularities.
  & we were stuck with not being able to handle other regulators.

Tiling formula: can use any regulator, nothing to do by hand.

Subtractions reduce to exactly the needed $\left(\frac{1}{\epsilon_{\rm UV}} - \frac{1}{\epsilon_{\rm IR}}\right)$ terms for dim.reg. setup.

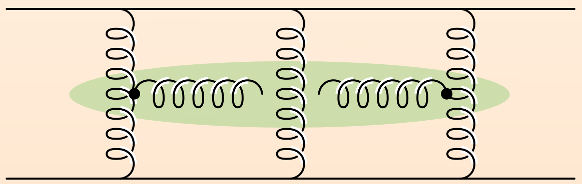
Non-Relativistic EFT (NRQCD, NRQED)

Unphysical singularities come from taking a double limit:

1) $k^\mu \gg E$, then $k^\mu \to 0$

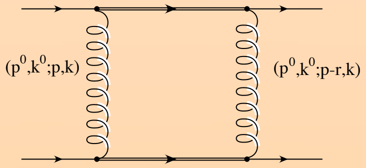
soft overlaps ultrasoft region

IR div. in QCD static potential

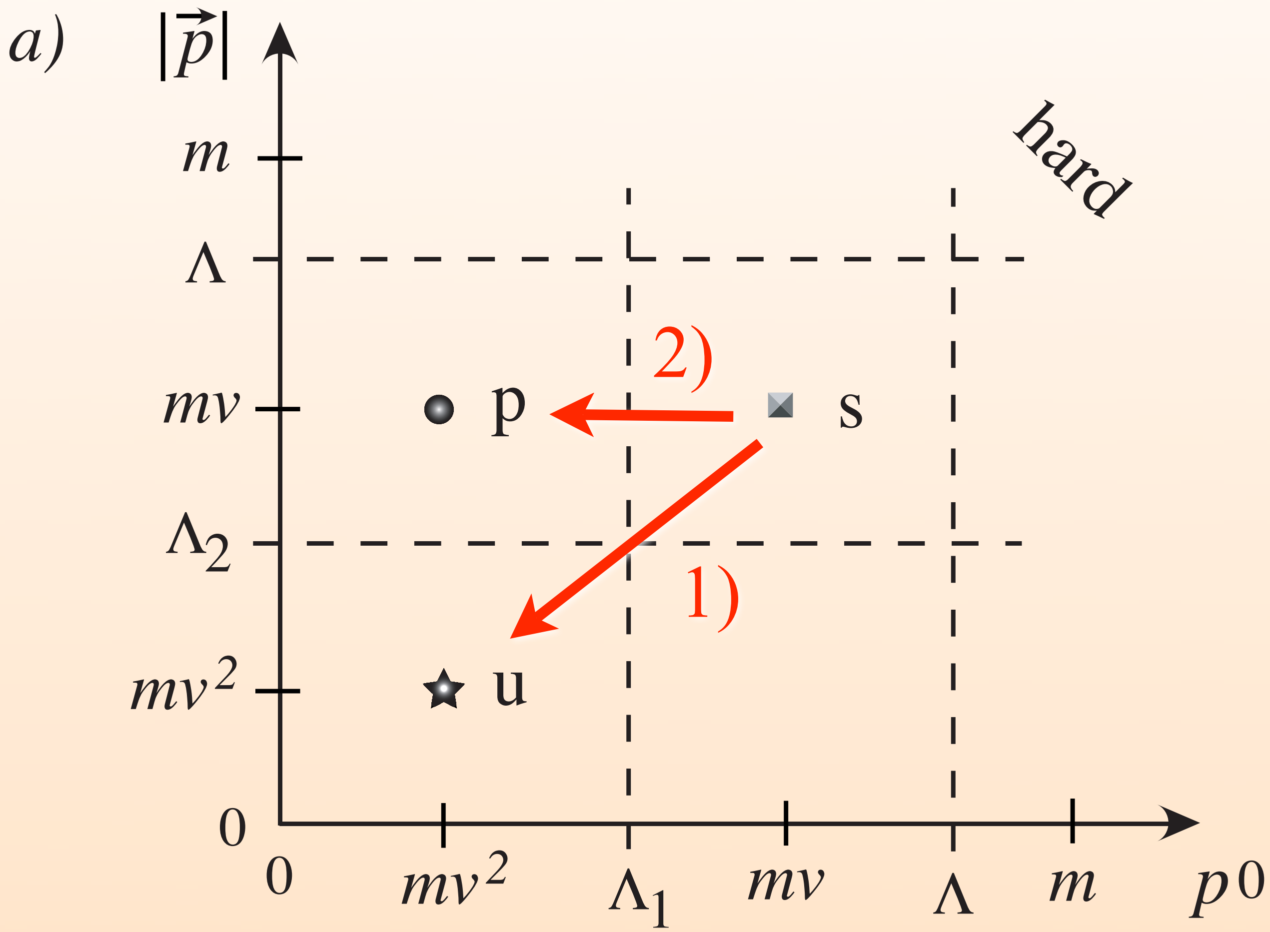


2) $k_0 \gg \dfrac{\mathbf{k}^2}{2m}$, then $k_0 \to 0$

soft overlaps potential region, pinch singularity



$$\int \frac{dk^0}{(k^0 + i0^+)(-k^0 + i0^+)} f(k^0)$$

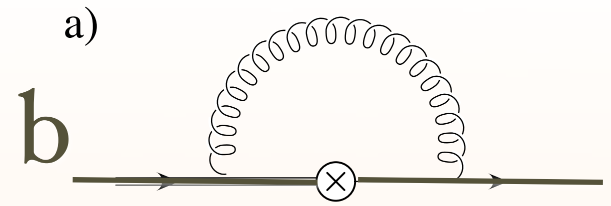


Non-singular with zero-bin.

A different momentum space mode describes the infrared in the region of the singularity.

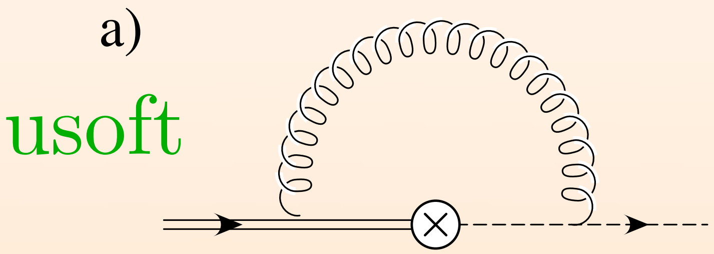
$\boxed{\text{QCD}}$

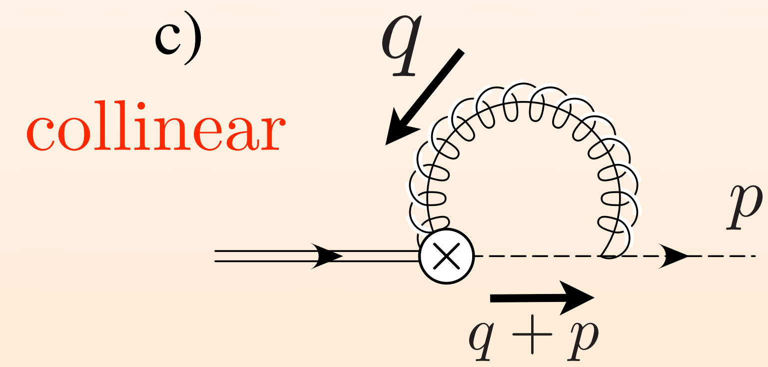
$$J^{\rm QCD} = \bar{s}\,\Gamma\, b$$

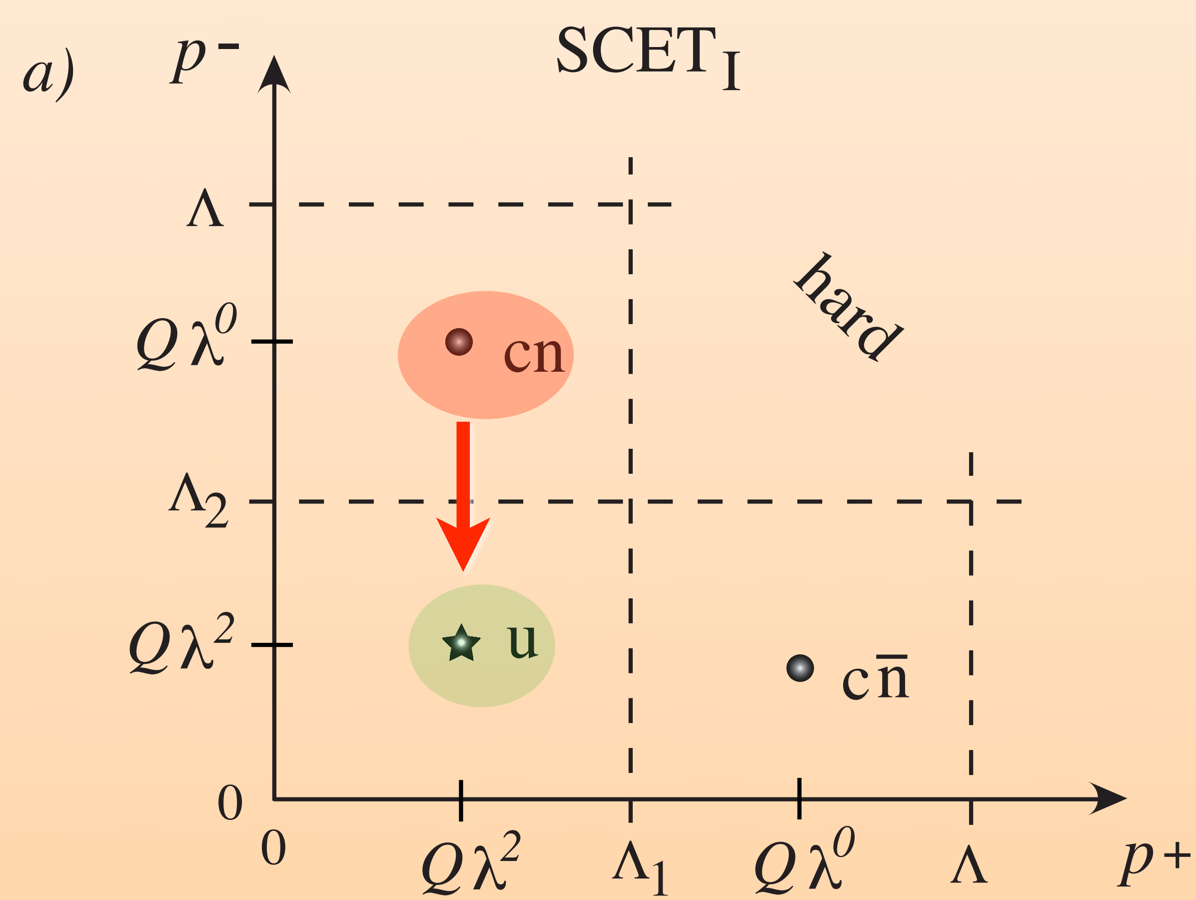


$\boxed{\text{SCET}_{\rm I}}$

$$J^{\rm SCET} = (\bar{\xi}_n W)_\omega \Gamma h_v$$

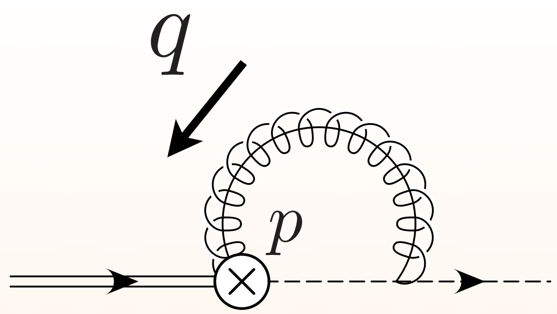






Apply to

c)



using dim.reg. in UV

$p^2 \neq 0$ in IR

avoids overcounting the usoft region

$$\sum_{q\neq 0,\, q\neq -p} \int \frac{d^4 q_r}{(2\pi)^4} \frac{2\bar{n}\cdot(q+p)}{(\bar{n}\cdot q + i0^+)((q+p)^2+i0^+)(q^2+i0^+)}$$

$$= \int \frac{d^d q}{(2\pi)^d}\left[\frac{2\bar{n}\cdot(q+p)}{(\bar{n}\cdot q+i0^+)[(q+p)^2+i0^+](q^2+i0^+)} - \underline{\frac{2\bar{n}\cdot p}{(\bar{n}\cdot q+i0^+)[n\cdot q\,\bar{n}\cdot p+p^2+i0^+](q^2+i0^+)}}\right]$$

subtraction

$$= -\frac{i}{16\pi^2}\left[-\frac{2}{\epsilon_{\rm IR}\epsilon_{\rm UV}} - \frac{2}{\epsilon_{\rm IR}}\ln\left(\frac{\mu^2}{-p^2}\right) - \ln^2\left(\frac{\mu^2}{-p^2}\right) + \left(\frac{2}{\epsilon_{\rm IR}} - \frac{2}{\epsilon_{\rm UV}}\right)\ln\left(\frac{\mu}{\bar{n}\cdot p}\right) + \ldots \right.$$

$$\left. - \underline{\left(\frac{2}{\epsilon_{\rm UV}} - \frac{2}{\epsilon_{\rm IR}}\right)\left\{\frac{1}{\epsilon_{\rm UV}} + \ln\left(\frac{\mu^2}{-p^2}\right) - \ln\left(\frac{\mu}{\bar{n}\cdot p}\right)\right\}}\right]$$

$$= -\frac{i}{16\pi^2}\left[-\frac{2}{\epsilon_{\rm UV}^2} - \frac{2}{\epsilon_{\rm UV}}\ln\left(\frac{\mu^2}{-p^2}\right) - \ln^2\left(\frac{\mu^2}{-p^2}\right)\right] + \ldots$$

- UV collinear singularity comes from $\bar{n}\cdot q \to \infty$ (in subtraction term)

  This is crucial for it to be independent of the choice of IR regulator.

  Divergences are removed by counterterms as usual.

eg. of another regulator

Cutoffs: $\Omega_\perp^2 \le \vec{q}_\perp^{\,2} \le \Lambda_\perp^2 \qquad \Omega_-^2 \le (q^-)^2 \le \Lambda_-^2$

no constraint on q^+, p^μ on-shell

QCD

$$I_{\rm full}^{b\to s\gamma} = \frac{i}{8\pi^2}\Big[\mathrm{Li}_2\Big(\frac{-\Omega_\perp^2}{\Omega_-^2}\Big) + \ln\Big(\frac{\Omega_-}{p^-}\Big)\ln\Big(\frac{\Omega_- p^-}{\Omega_\perp^2}\Big)\Big] + \ldots$$

SCET

$$I_{\rm us}^{b\to s\gamma} = \frac{i}{8\pi^2}\Big[\mathrm{Li}_2\Big(\frac{-\Omega_\perp^2}{\Omega_-^2}\Big) + \ln\Big(\frac{\Omega_-}{\Lambda_-}\Big)\ln\Big(\frac{\Omega_-\Lambda_-}{\Omega_\perp^2}\Big)\Big]$$

$$I_{\rm C}^{b\to s\gamma} = \frac{i}{8\pi^2}\Big[-\ln\Big(\frac{\Omega_\perp^2}{\Lambda_\perp^2}\Big)\ln\Big(\frac{\Omega_-}{p^-}\Big)\Big] - \underline{\frac{i}{8\pi^2}\Big[-\ln\Big(\frac{\Omega_\perp^2}{\Lambda_\perp^2}\Big)\ln\Big(\frac{\Omega_-}{\Lambda_-}\Big)\Big]} = \frac{i}{8\pi^2}\Big[-\ln\Big(\frac{\Omega_\perp^2}{\Lambda_\perp^2}\Big)\ln\Big(\frac{\Lambda_-}{p^-}\Big)\Big] + \ldots$$

$$I_{\rm us}^{b\to s\gamma} + I_{\rm C}^{b\to s\gamma} = \frac{i}{8\pi^2}\Big[\mathrm{Li}_2\Big(\frac{-\Omega_\perp^2}{\Omega_-^2}\Big) + \ln\Big(\frac{\Omega_-}{p^-}\Big)\ln\Big(\frac{\Omega_- p^-}{\Omega_\perp^2}\Big) + \ln^2\Big(\frac{\Lambda_\perp}{p^-}\Big) - \ln^2\Big(\frac{\Lambda_\perp}{\Lambda_-}\Big)\Big] + \ldots$$

IR matches again,
zero-bin subtraction is crucial.

$\mathrm{SCET}_{\mathrm{II}}$

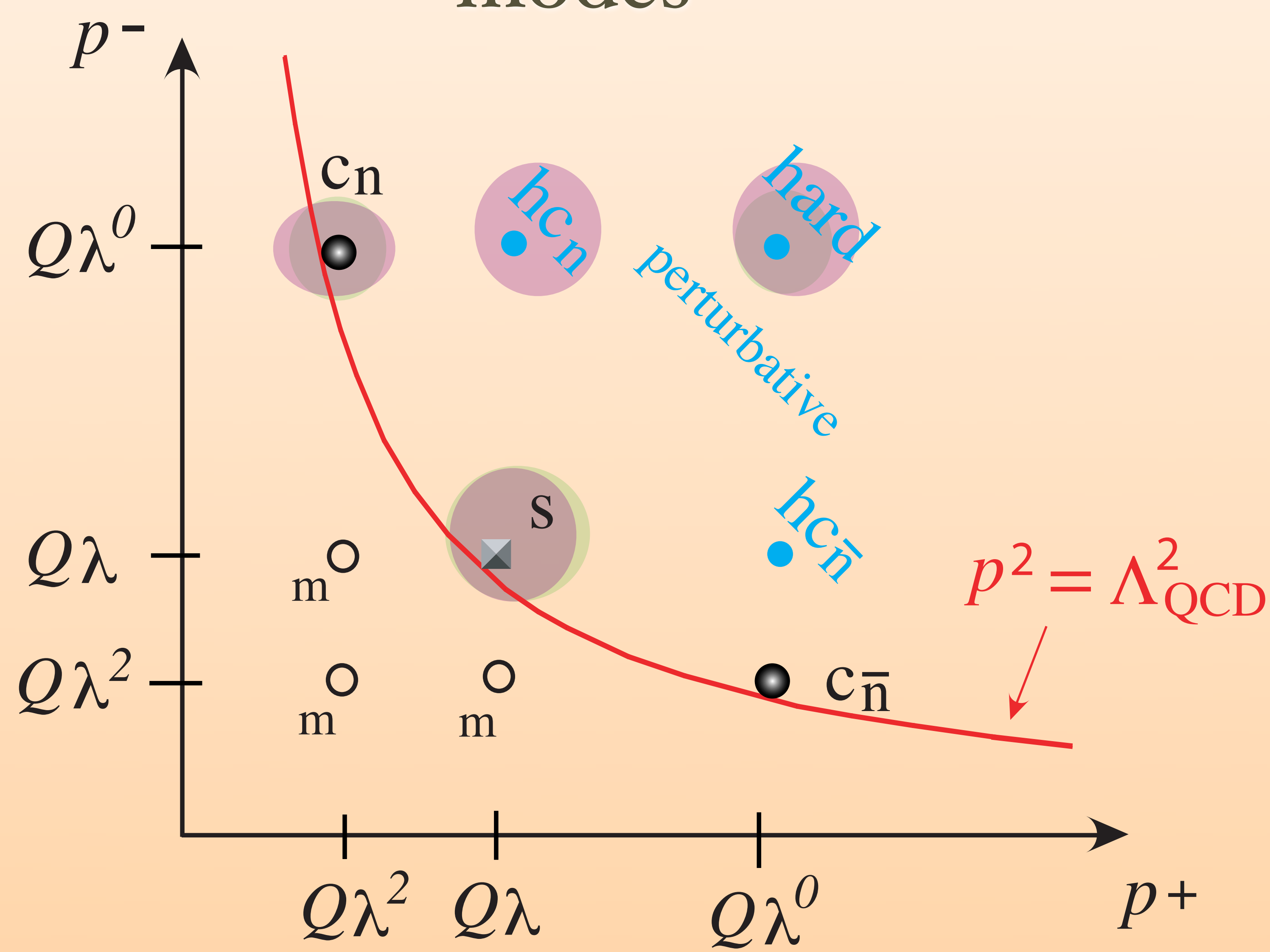
$$\lambda = \frac{\Lambda}{Q}$$

- all known examples of endpoint singularities have > one hadron
- $\mathrm{SCET}_{\mathrm{II}}$ allows us to treat cases with two or more hadrons

  eg. $B \to D\pi$, $B \to \pi\ell\bar{\nu}$, $e^- p \to e^- X\,\pi$

- $c_n, s, c_{\bar{n}}$ are definitely required as low energy modes

modes



SCET$_{\rm II}$

$$\lambda = \frac{\Lambda}{Q}$$

- all known examples of endpoint singularities have > one hadron
- SCET$_{\rm II}$ allows us to treat cases with two or more hadrons
  eg. $B \to D\pi$, $B \to \pi\ell\bar{\nu}$, $e^- p \to e^- X \pi$
- $c_n, s, c_{\bar{n}}$ are definitely required as low energy modes
- "messenger" scales ○

show up in perturbation theory

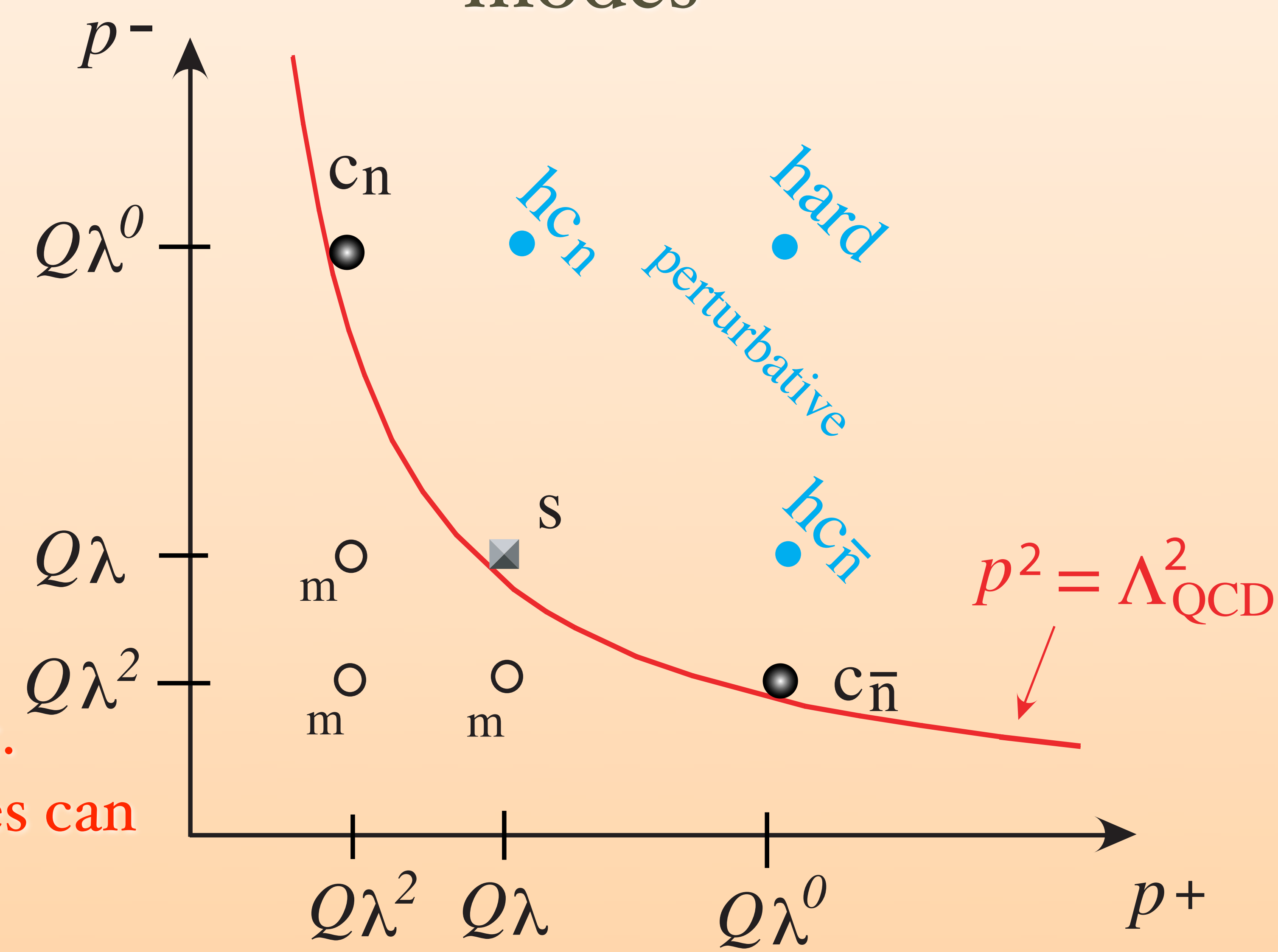
Becher, Hill, Neubert

but only for certain IR regulators

Beneke, Feldmann; Bauer, Dorsten, Salem

Must consider effect of confinement.
We will see shortly that the ○ modes can be absorbed into the other d.o.f.

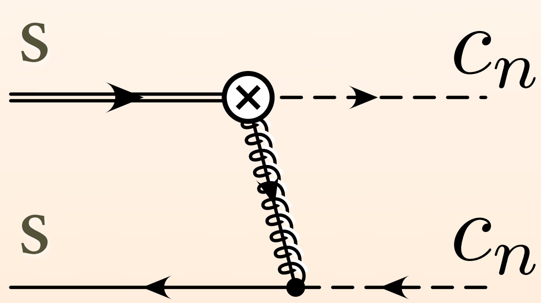
modes



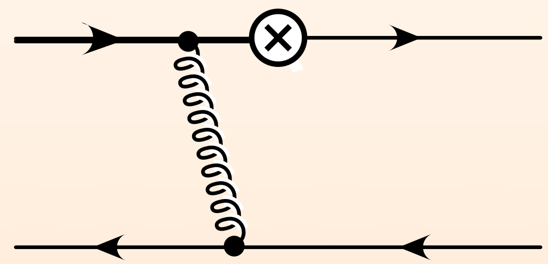
For our endpoint divergence

$$\int_0^1 \mathrm{d}x \, \frac{\phi_\pi(x)}{x^2} \, ,$$ the singularity comes from taking a double limit:

collinear $k^- \gg k^\perp, k^+$, then $k^- \to 0$



in QCD was



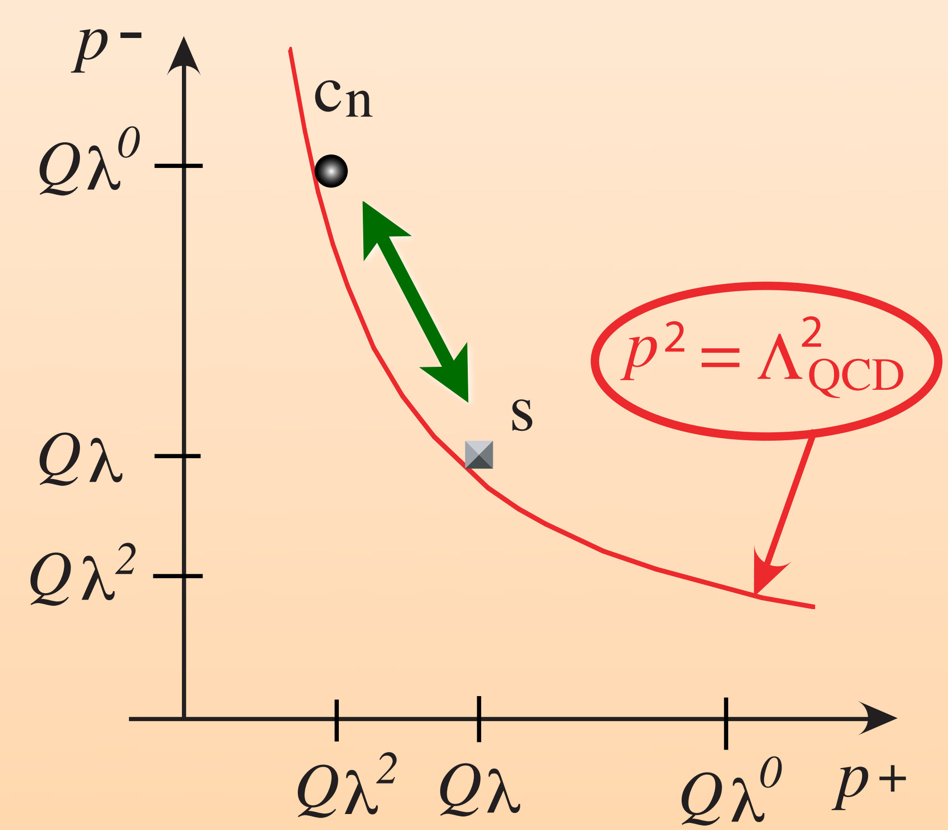
and $k^- \to 0$ encounters the soft region where there is another mode

Based on our experience the formula:

$$\sum_{p_1 \neq 0} \int dp_{1r} \, F^{(q_b)}(p_1) = \int dp_1 \Big[F^{(q_b)}(p_1) - F_{\mathrm{subt}}^{(q_b \to q_a)}(p_1) \Big]$$

should avoid double counting the soft region, and thus remove the singularities here too.

Note: absence of onshell modes between c_n and s is due to a rapidity gap.

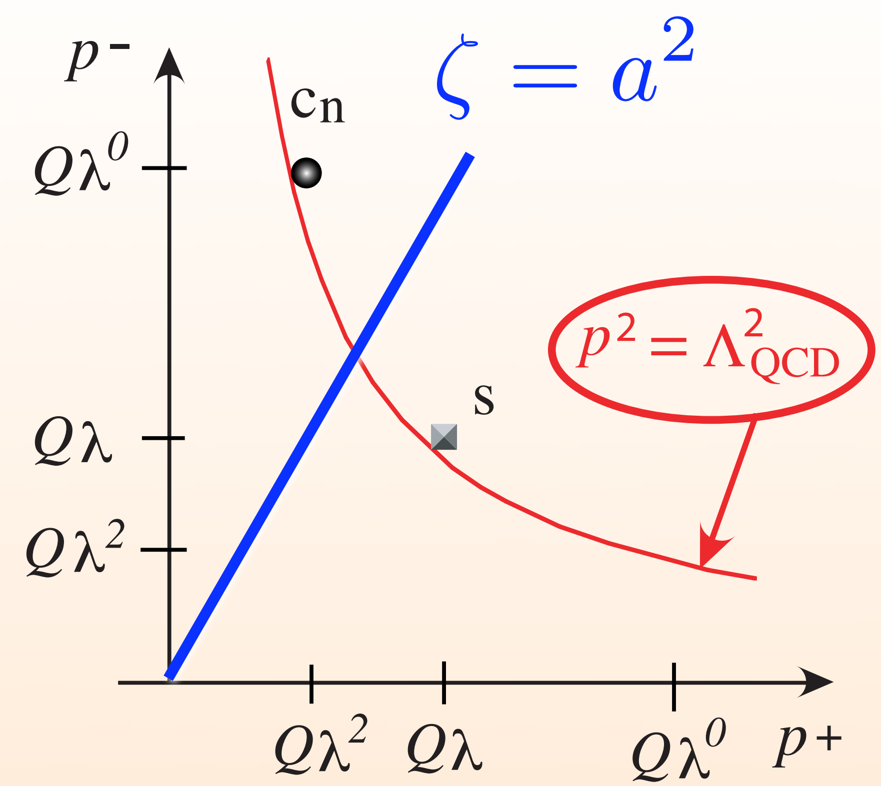


Rapidity distinguishes the d.o.f.

$$\zeta_k = k^- / k^+$$

n-collinear : $\zeta_p \sim \lambda^{-2} \gg 1$

soft: $\zeta_p \sim \lambda^0 \sim 1$



Check how this works with a Wilsonian rapidity cutoff (zero-bin subtraction = 0)

Wick rotated rapidity: soft: $-a^2 \le \zeta_k' \le a^2\,,$

collinear: $-a^2 \ge \zeta_k' \quad \text{or} \quad \zeta_k' \ge a^2$

toy example

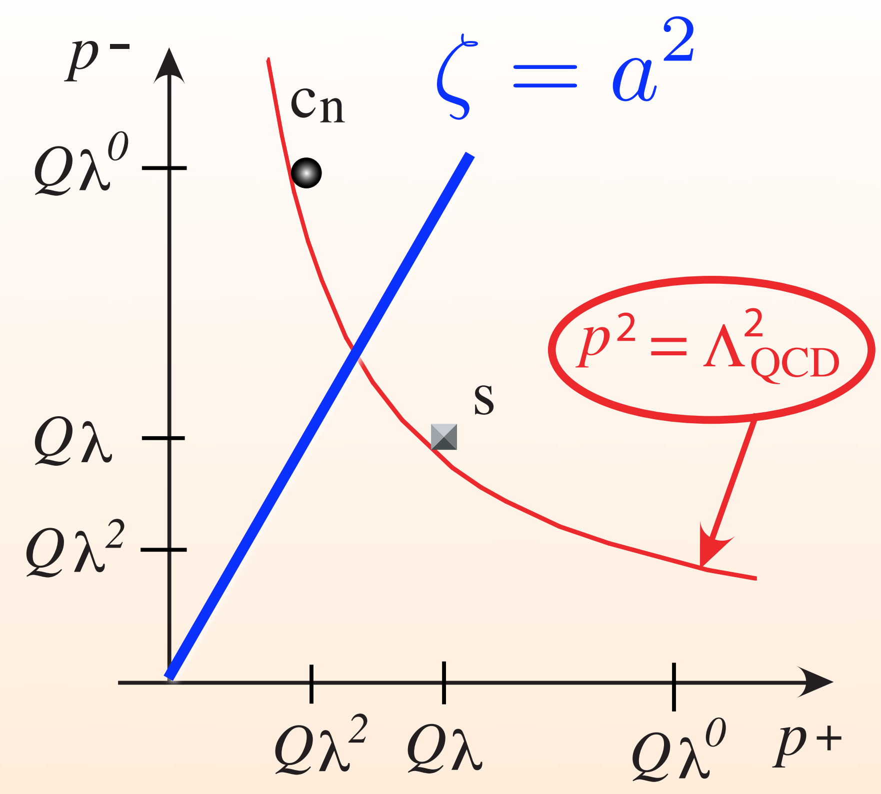
$$I_{\rm full}^{\rm scalar} = \int \frac{d^Dk}{(2\pi)^D} \frac{1}{[(k-\ell)^2+i0^+][k^2+i0^+][(k-p)^2+i0^+]} = \frac{-i}{16\pi^2(p^-\ell^+)}\left[\frac{1}{\epsilon_{\rm IR}^2} - \frac{1}{\epsilon_{\rm IR}}\ln\left(\frac{p^-\ell^+}{\mu^2}\right) + \frac{1}{2}\ln^2\left(\frac{p^-\ell^+}{\mu^2}\right) - \frac{\pi^2}{12}\right]$$

$$I_s = \int \frac{\mathrm{d}^Dk}{(2\pi)^D} \frac{1}{-p^-k^+ + i0^+} \frac{1}{k^+k^- - \mathbf{k}_\perp^2 + i0^+} \frac{1}{k^+k^- - k^-\ell^+ - \mathbf{k}_\perp^2 + i0^+}$$

$$I_c = \int \frac{\mathrm{d}^Dk}{(2\pi)^D} \frac{1}{-k^-\ell^+ + i0^+} \frac{1}{k^+k^- - \mathbf{k}_\perp^2 + i0^+} \frac{1}{k^+k^- - k^+p^- - \mathbf{k}_\perp^2 + i0^+}$$

Rapidity distinguishes the d.o.f.

$$\zeta_k = k^-/k^+$$



n-collinear : $\quad \zeta_p \sim \lambda^{-2} \gg 1$

soft: $\quad \zeta_p \sim \lambda^0 \sim 1$

Check how this works with a Wilsonian rapidity cutoff (zero-bin subtraction = 0)

Wick rotated rapidity: soft: $\quad -a^2 \leq \zeta'_k \leq a^2\,,$

collinear: $\quad -a^2 \geq \zeta'_k \quad \text{or} \quad \zeta'_k \geq a^2$

toy example

$$I^{\rm scalar}_{\rm full} = \int \frac{d^Dk}{(2\pi)^D} \frac{1}{[(k-\ell)^2+i0^+][k^2+i0^+][(k-p)^2+i0^+]} = \frac{-i}{16\pi^2(p^-\ell^+)}\left[\frac{1}{\epsilon^2_{\rm IR}} - \frac{1}{\epsilon_{\rm IR}}\ln\left(\frac{p^-\ell^+}{\mu^2}\right) + \frac{1}{2}\ln^2\left(\frac{p^-\ell^+}{\mu^2}\right) - \frac{\pi^2}{12}\right]$$

$$I^{\rm scalar}_{\rm soft} = \frac{-i}{16\pi^2(p^-\ell^+)}\left[\frac{1}{2\epsilon^2_{\rm IR}} - \frac{1}{\epsilon_{\rm IR}}\ln\left(\frac{\ell^+}{\mu_+}\right) + \ln^2\left(\frac{\ell^+}{\mu_+}\right) - \frac{\pi^2}{16}\right] \qquad \mu_+ = \mu/a.$$

$$I^{\rm scalar}_{\rm cn} = \frac{-i}{16\pi^2(p^-\ell^+)}\left[\frac{1}{2\epsilon^2_{\rm IR}} - \frac{1}{\epsilon_{\rm IR}}\ln\left(\frac{p^-}{\mu_-}\right) + \ln^2\left(\frac{p^-}{\mu_-}\right) - \frac{\pi^2}{16}\right] \qquad \mu_- = a\mu.$$

$$\boxed{\mu_+\mu_- = \mu^2}$$

IR reproduced

$$I^{\rm scalar}_{\rm matching} = \frac{-i}{16\pi^2(p^-\ell^+)}\left[-\frac{1}{2}\ln^2\left(\frac{p^-\mu_+}{\mu_-\,\ell^+}\right) + \frac{\pi^2}{24}\right]$$

The zero-bin minimal subtractions can be used to handle the overlaps in dim. reg. This ensures soft does not overlap collinear and visa versa.

Use democratic subtractions to avoid introducing hard cutoff scale

However, these subtractions produce problems with rapidity divergences in the UV. Standard dimensional regularization does not suffice for these.

eg.

from scaling into collinear region

$$I_{\rm soft}^{\rm scalar} = \sum_{k^+\neq 0} \int \frac{d^D k_r}{(2\pi)^D} \frac{\mu^{2\epsilon}}{[k^2-\ell^+ k^- + i0^+][k^2-m^2+i0^+][-p^- k^+ + i0^+]}$$

$$= \int \frac{d^D k}{(2\pi)^D} \frac{\mu^{2\epsilon}}{[k^2-\ell^+ k^- + i0^+][k^2-m^2+i0^+][-p^- k^+ + i0^+]} - \frac{\mu^{2\epsilon}}{[-\ell^+ k^- + i0^+][k^2-m^2+i0^+][-p^- k^+ + i0^+]}$$

$$= \frac{-i\,\Gamma(\epsilon)\,\mu^{2\epsilon}}{16\pi^2(p^-\ell^+)} \left[\int_0^{\ell^+} \frac{dk^+}{k^+} \left[\frac{(\ell^+ - k^+)m^2}{\ell^+}\right]^{-\epsilon} - \int_0^{\infty} \frac{dk^+}{k^+} \, (m^2)^{-\epsilon} \right]$$

Lets invent a gauge invariant dim.reg. like regulator that is formulated at the level of operators:

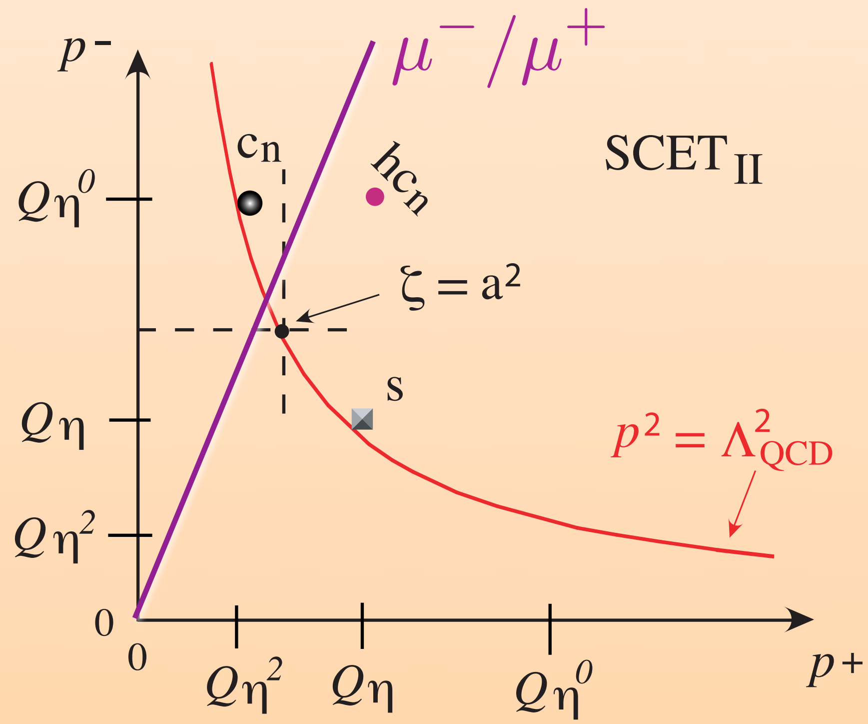
$$\boxed{\mu_+ \, \mu_- = \mu^2}$$

$$J(p_j^-, k_j^+) \Big[(\bar{q}_s S)_{k_1^+} \Gamma_s (S^\dagger q_s)_{k_2^+} \Big] \Big[(\bar{\xi}_n W)_{p_1^-} \Gamma_n (W^\dagger \xi_n)_{p_2^-} \Big]$$

$$\xrightarrow{\text{dim.reg.}} J(p_j^-, k_j^+, \mu_\pm)\, \mu^{2\epsilon} \bigg[(\bar{q}_s S)_{k_1^+} \frac{|\mathcal{P}^\dagger|^\epsilon}{\mu_+^\epsilon} \Gamma_s \frac{|\mathcal{P}|^\epsilon}{\mu_+^\epsilon} (S^\dagger q_s)_{k_2^+} \bigg] \bigg[(\bar{\xi}_n W)_{p_1^-} \frac{|\bar{\mathcal{P}}^\dagger|^\epsilon}{\mu_-^\epsilon} \Gamma_n \frac{|\bar{\mathcal{P}}|^\epsilon}{\mu_-^\epsilon} (W^\dagger \xi_n)_{p_2^-} \bigg]$$

$$= J(p_j^-, k_j^+, \mu_\pm, \mu^2)\, \mu^{2\epsilon} \bigg[\frac{|k_1^+ k_2^+|^\epsilon}{\mu_+^{2\epsilon}} (\bar{q}_s S)_{k_1^+} \Gamma_s (S^\dagger q_s)_{k_2^+} \bigg] \bigg[\frac{|p_1^- p_2^-|^\epsilon}{\mu_-^{2\epsilon}} (\bar{\xi}_n W)_{p_1^-} \Gamma_n (W^\dagger \xi_n)_{p_2^-} \bigg]$$

regulates UV rapidity divergences



(Using absolute values preserves analyticity, it corresponds to positive mom. of particles and anti-particles.)

Lets invent a gauge invariant dim.reg. like regulator that is formulated at the level of operators:

$$\boxed{\mu_+ \, \mu_- = \mu^2}$$

$$J(p_j^-, k_j^+) \big[(\bar{q}_s S)_{k_1^+} \Gamma_s (S^\dagger q_s)_{k_2^+} \big] \big[(\bar{\xi}_n W)_{p_1^-} \Gamma_n (W^\dagger \xi_n)_{p_2^-} \big]$$

$$\xrightarrow{\text{dim.reg.}} J(p_j^-, k_j^+, \mu_\pm)\, \mu^{2\epsilon} \bigg[(\bar{q}_s S)_{k_1^+} \frac{|\mathcal{P}^\dagger|^\epsilon}{\mu_+^\epsilon} \Gamma_s \frac{|\mathcal{P}|^\epsilon}{\mu_+^\epsilon} (S^\dagger q_s)_{k_2^+} \bigg] \bigg[(\bar{\xi}_n W)_{p_1^-} \frac{|\bar{\mathcal{P}}^\dagger|^\epsilon}{\mu_-^\epsilon} \Gamma_n \frac{|\bar{\mathcal{P}}|^\epsilon}{\mu_-^\epsilon} (W^\dagger \xi_n)_{p_2^-} \bigg]$$

$$= J(p_j^-, k_j^+, \mu_\pm, \mu^2)\, \mu^{2\epsilon} \bigg[\frac{|k_1^+ k_2^+|^\epsilon}{\mu_+^{2\epsilon}} (\bar{q}_s S)_{k_1^+} \Gamma_s (S^\dagger q_s)_{k_2^+} \bigg] \bigg[\frac{|p_1^- p_2^-|^\epsilon}{\mu_-^{2\epsilon}} (\bar{\xi}_n W)_{p_1^-} \Gamma_n (W^\dagger \xi_n)_{p_2^-} \bigg] \quad + \ldots$$

Note that the factors should be interpreted as in the matrix element

to give $\mu_\pm$ dependent distribution functions. Expect something like:

$$\int dk^+ dp^- \; J(k^+, p^-, \mu_+, \mu_-)\, \phi_n(p^-, \mu_-, \mu^2)\, \phi_s(k^+, \mu_+, \mu^2)$$

An additional ingredient is needed as indicated by the $+\ldots$

Lets try it out

Three IR Masses, $\mathbf{m_1}$, $\mathbf{m_2}$, $\mathbf{m_3}$

$\ell =$ soft momentum

$p =$ collinear momentum

$$I^{\rm scalar}_{\rm full} = \int \frac{d^Dk}{(2\pi)^D} \frac{1}{[(k-\ell)^2 - m_2^2 + i0^+][k^2 - m_1^2 + i0^+][(k-p)^2 - m_3^2 + i0^+]}$$

$$= \frac{-i}{16\pi^2(p^-\ell^+)} \left[\frac{1}{2}\ln^2\left(\frac{m_1^2}{p^-\ell^+}\right) + {\rm Li}_2\left(1 - \frac{m_2^2}{m_1^2}\right) + {\rm Li}_2\left(1 - \frac{m_3^2}{m_1^2}\right)\right].$$

$$I^{\rm scalar}_{\rm soft} = \sum_{k^+\neq 0} \int \frac{d^Dk_r}{(2\pi)^D} \frac{\mu^{2\epsilon}}{[k^2 - \ell^+ k^- - m_2^2 + i0^+][k^2 - m_1^2 + i0^+][-p^- k^+ + i0^+]} \; \frac{|k^+|^\epsilon |k^+ - \ell^+|^\epsilon}{\mu_+^{2\epsilon}}$$

$$I^{\rm scalar}_{\rm cn} = \sum_{k^-\neq 0} \int \frac{d^Dk'_r}{(2\pi)^D} \frac{\mu^{2\epsilon}}{[-\ell^+ k^- + i0^+][k^2 - m_1^2 + i0^+][k^2 - p^- k^+ - m_3^2 + i0^+]} \; \frac{|k^-|^\epsilon |k^- - p^-|^\epsilon}{\mu_-^{2\epsilon}}$$

$k^+ \to 0$ cancels

$$I^{\rm scalar}_{\rm soft} = \frac{-i\,\Gamma(\epsilon)\,\mu^{2\epsilon}}{16\pi^2(p^-\ell^+)} \left[\int_0^{\ell^+} \frac{dk^+}{k^+} \left[m_1^2\left(1 - \frac{k^+}{\ell^+}\right) + m_2^2 \frac{k^+}{\ell^+}\right]^{-\epsilon} \left|\frac{k^+(k^+ - \ell^+)}{{\mu_+}^2}\right|^\epsilon - \int_0^\infty \frac{dk^+}{k^+} \left[m_1^2\right]^{-\epsilon} \left|\frac{k^+(k^+ - \ell^+)}{{\mu_+}^2}\right|^\epsilon \right]$$

$$= \frac{-i}{16\pi^2(p^-\ell^+)} \left[\frac{1}{2\epsilon_{\rm UV}^2} + \frac{1}{\epsilon_{\rm UV}} \ln\left(\frac{\ell^+}{\mu_+}\right) - \frac{1}{2\epsilon_{\rm UV}} \ln\left(\frac{m_1^2}{\mu^2}\right) + \ln^2\left(\frac{\ell^+}{\mu_+}\right) + \frac{5\pi^2}{24} + \frac{1}{4}\ln^2\left(\frac{m_1^2}{\mu^2}\right) - \ln\left(\frac{m_1^2}{\mu^2}\right)\ln\left(\frac{\ell^+}{\mu_+}\right) + {\rm Li}_2\left(1 - \frac{m_2^2}{m_1^2}\right)\right].$$

$k^+ \to \infty$ regulated

looks good! except for $\frac{1}{2\epsilon_{\rm UV}} \ln\left(\frac{m_1^2}{\mu^2}\right)$ term, which had no analog for the Wilsonian rapidity regulator:

+... is $+\, O_D$, with a counterterm coefficient which cancels this term.

Three IR Masses, $\mathbf{m_1}$, $\mathbf{m_2}$, $\mathbf{m_3}$

$$I^{\rm scalar}_{\rm full} = \int \frac{d^D k}{(2\pi)^D} \frac{1}{[(k-\ell)^2 - m_2^2 + i0^+][k^2 - m_1^2 + i0^+][(k-p)^2 - m_3^2 + i0^+]}$$
$$= \frac{-i}{16\pi^2 (p^-\ell^+)} \Big[\frac{1}{2} \ln^2\Big(\frac{m_1^2}{p^-\ell^+}\Big) + {\rm Li}_2\Big(1 - \frac{m_2^2}{m_1^2}\Big) + {\rm Li}_2\Big(1 - \frac{m_3^2}{m_1^2}\Big)\Big].$$

IR matches

renormalized

$$I^{\rm scalar}_{\rm soft+cn} = \frac{-i}{16\pi^2 (p^-\ell^+)} \Big[\frac{1}{2} \ln^2\Big(\frac{m_1^2}{p^-\ell^+}\Big) + {\rm Li}_2\Big(1 - \frac{m_2^2}{m_1^2}\Big) + {\rm Li}_2\Big(1 - \frac{m_3^2}{m_1^2}\Big) - \ln\Big(\frac{m_1^2}{\mu^2}\Big) \ln\Big(\frac{\mu^2}{\mu_-\mu_+}\Big)$$
$$+ \ln^2\Big(\frac{p^-}{\mu_-}\Big) + \ln^2\Big(\frac{\ell^+}{\mu_+}\Big) - \frac{1}{2} \ln^2\Big(\frac{p^-\ell^+}{\mu^2}\Big) + \frac{5\pi^2}{12}\Big].$$

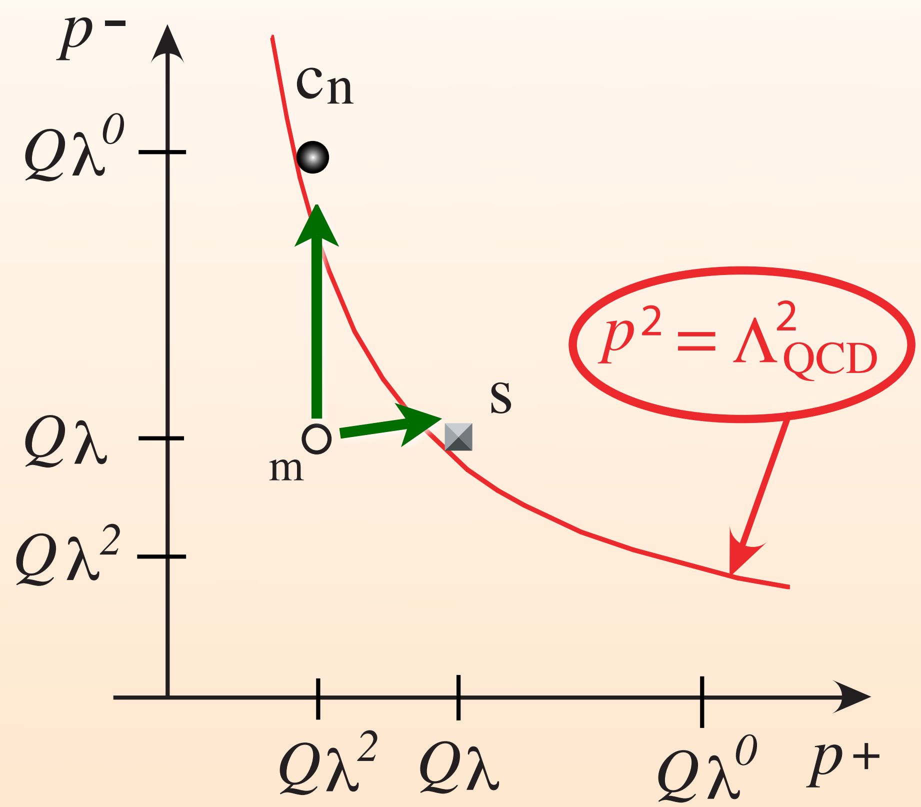
vanishes for

$\mu^+\mu^- = \mu^2$

$$I^{\rm scalar}_{\rm match} = \frac{-i}{16\pi^2 (p^-\ell^+)} \Big[- \frac{1}{2} \ln^2\Big(\frac{p^-\mu_+}{\mu_-\ell^+}\Big) - \frac{5\pi^2}{12}\Big]$$

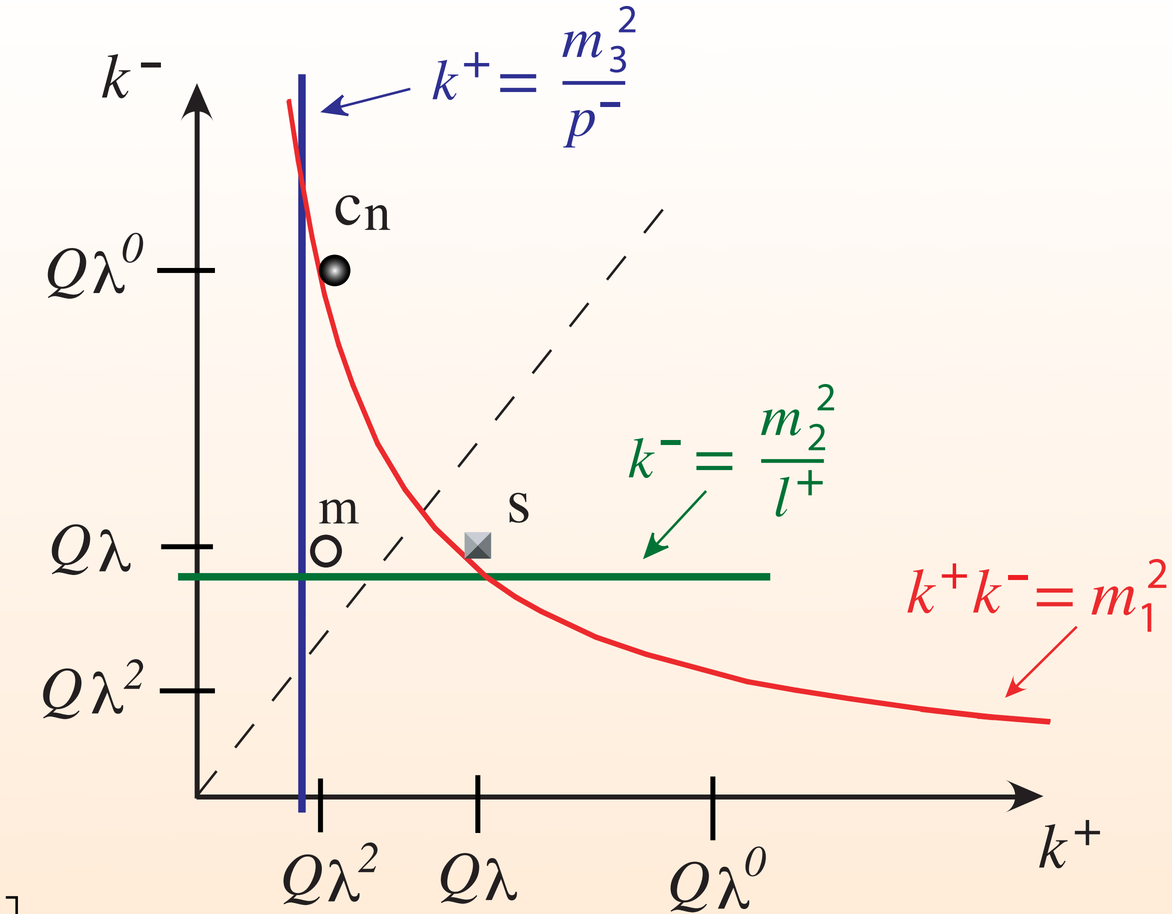
What about the messenger modes?

Messenger is absorbed into a combination of c_n and s



Lets see how

Three IR Masses, $\mathbf{m_1, m_2, m_3}$



study double log

What if $m_1 = 0$?

$$I_{\rm full}^{\rm scalar} = \frac{-i}{16\pi^2(p^-\ell^+)}\bigg\{\frac{1}{2}\ln^2\Big[\frac{\xi - i0^+}{Q^4}\Big] + {\rm Li}_2\Big[\frac{Q^2(m_1^2-m_2^2)}{\xi} - i0^+\Big]$$
$$+{\rm Li}_2\Big[\frac{Q^2(m_1^2-m_3^2)}{\xi} - i0^+\Big] - {\rm Li}_2\Big[\frac{-(m_1^2-m_2^2)(m_1^2-m_3^2)}{\xi}\Big]\bigg\}$$

$$\xi \equiv Q^2 m_1^2 - m_2^2 m_3^2$$

$$I_{\rm full}^{\rm scalar}(m_1=0) = \frac{-i}{16\pi^2(p^-\ell^+)}\Big[\ln\Big(\frac{m_2^2}{Q^2}\Big)\ln\Big(\frac{m_3^2}{Q^2}\Big)\Big]$$

If we set $m_1 = 0$ we become sensitive to “m” region.

In QCD we expect confinement to introduce a scale like $m_1 \neq 0$

Then the s & c modes absorb “m”, just as we saw in our calculations with rapidity regulators.

Implications for singular Convolutions

Start by ignoring invariant mass UV renormalization

$$A_\pi = \sum_{p^-_{1,2}\neq 0} \int dp^-_{1r} dp^-_{2r}\ J(p_1^-, p_2^-)\ \langle \pi_n(p_\pi) \big| (\bar\xi_n W)_{p_1^-} \bar{n}\!\!\!/ \gamma_5 (W^\dagger \xi_n)_{-p_2^-} \big| 0 \rangle \left| \frac{p_1^- p_2^-}{\mu_-^2} \right|^\epsilon$$

$$J(x_1, x_2) = 1/(p_1^-)^2 = 1/[(\bar n\!\cdot\! p_\pi)^2\, x_1^2]$$

$$= -i \frac{f_\pi}{\bar n\!\cdot\! p_\pi} \Big(\frac{\bar n\!\cdot\! p_\pi}{\mu_-}\Big)^{2\epsilon} \sum_{x_1\neq 0} \int dx_{1r}\, dx_2\ \frac{1}{(x_1)^2}\ \delta(1-x_1-x_2)\ \phi_\pi(x_1, x_2)\ \big|x_1 x_2\big|^\epsilon$$

$$= -i \frac{f_\pi}{\bar n\!\cdot\! p_\pi} \Big(\frac{\bar n\!\cdot\! p_\pi}{\mu_-}\Big)^{2\epsilon} \sum_{x_1\neq 0} \int dx_{1r}\ \frac{1}{(x_1)^2}\ \theta(1-x_1)\theta(x_1)\ \hat\phi_\pi(x_1)\ \big|x_1(1-x_1)\big|^\epsilon$$

$$= \frac{-i f_\pi}{\bar n\!\cdot\! p_\pi} \Big(\frac{p_\pi^-}{\mu_-}\Big)^{2\epsilon} \int dx_1\ \frac{\theta(x_1)}{(x_1)^2} \Big[\theta(1-x_1)\ \hat\phi_\pi(x_1) - x_1 \hat\phi'_\pi(0)\Big] \big|x_1(1-x_1)\big|^{-\epsilon}$$

$$= -i \frac{f_\pi}{\bar n\!\cdot\! p_\pi} \Big(\frac{\bar n\!\cdot\! p_\pi}{\mu_-}\Big)^{2\epsilon} \Big\{ \int_0^1 dx_1\ \frac{\phi_\pi(x_1) - x_1 \phi'_\pi(0)}{(x_1)^2} + \frac{1}{2\epsilon_{\rm UV}} \big[\phi'_\pi(0)\big] \Big\}$$

add $\quad O_{ct}^{[1]} = \int dp_2^- \Big[\frac{d}{dp_1^-}\Big] (\bar\xi_n W)_{p_1^-} \bar{n}\!\!\!/ \gamma_5 (W^\dagger \xi_n)_{-p_2^-} \Big|_{p_1^- \to 0}.$

$$A_\pi + A_\pi^{ct} = -i \frac{f_\pi}{\bar n\!\cdot\! p_\pi} \Big\{ \int_0^1 dx_1\ \frac{\phi_\pi(x_1,\mu) - x_1 \phi'_\pi(0,\mu)}{(x_1)^2} + \phi'_\pi(0,\mu)\ \ln\Big(\frac{\bar n\!\cdot\! p_\pi}{\mu_-}\Big) \Big\} + D(\mu, \mu_-)\ \phi'_\pi(0,\mu)$$

$$= \text{finite} = -i \frac{f_\pi}{\bar n\!\cdot\! p_\pi} \int_0^1 dx_1\ \frac{\phi_\pi(x_1)}{(x_1^2)_\varnothing}$$

Implications for singular Convolutions

Now realistic case, with invariant mass UV renormalization

all $\phi_\pi(x) \to \phi_\pi^\epsilon(x)$ UV renormalized, but now already a distribution in ϵ

In this case we find:

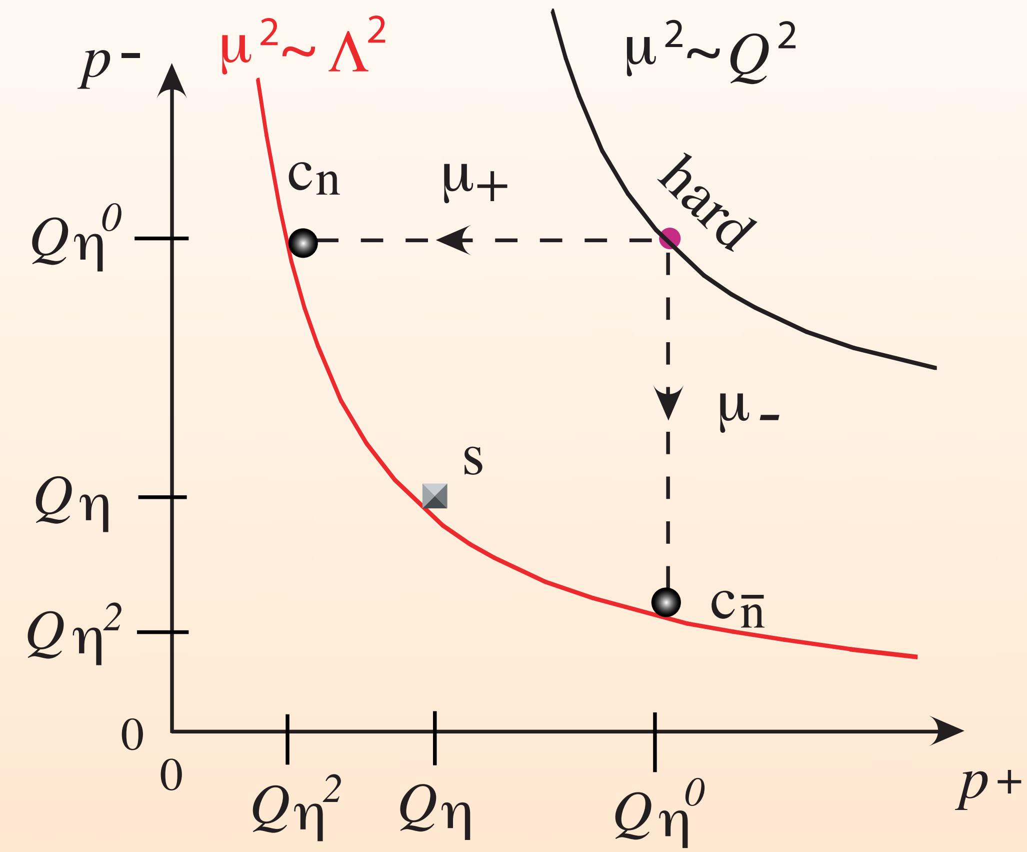
$$A_\pi = -i\frac{f_\pi}{\bar{n}\cdot p_\pi}\int_0^1 dx\left[\left(\frac{C(x,\mu,\mu_-,p_-)}{x^2}\right)_+ - \delta'(x)d(\mu,\mu_-,p_-)\right]\psi^\epsilon\left(x,\mu,\frac{\mu_-}{p^-}\right)$$

new distribution,
but same Brodsky-Lepage anom.dim.

- at lowest order μ_- dependence cancels between d & ψ^ϵ
- μ dependence causes mixing between C & d terms, but they close under RGE.
- preliminary, we are still performing cross-checks

RGE flow

$$\mu_+ \mu_- = \mu^2$$



Consider

$$A_\pi = \tilde{d}(\mu)\tilde{\phi}'(0,\mu) + \int_0^1 dx \left[C(x,\mu)\right]_+ \tilde{\phi}(x,\mu)$$

$$\mu \frac{d}{d\mu}\tilde{\phi}(x,\mu) \;=\; \int dy\, \gamma(x,y)\, \tilde{\phi}(y,\mu)$$

B.L. anom.dim.

$$\gamma(x,y) \;=\; \frac{C_F \alpha_s}{\pi}\left[\frac{x}{y}\Big(\frac{1}{y-x}+1\Big)\theta(y-x)\theta(x) + \frac{1-x}{1-y}\Big(\frac{1}{x-y}+1\Big)\theta(x-y)\theta(1-x)\right]_\oplus$$

Separate into 2 equations: C mixes into d and visa-versa

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\,\tilde{d}(\mu,\mu_0^-) \;=\; -\int_0^1 dx\; xC(x,\mu)\Big[\frac{\Gamma(x)}{x} - \Gamma^{(1)}(0)\Big] - \tilde{d}(\mu,\mu_0^-)\,\Gamma^{(1)}(0)\,,$$

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\Big[yC(y,\mu)\Big]_+ \;=\; -\int_0^1 dx\; xC(x,\mu)\,\Big[\frac{y}{x}\gamma(x,y) - y\gamma^{(1,0)}(0,y)\Big]_+ - \tilde{d}(\mu,\mu_0^-)\Big[y\gamma^{(1,0)}(0,y)\Big]_+$$

careful:

- distribution for $x = y$
- plus function for $y = 0$
- vanishes as x, as $x \to 0$

$$\gamma^{(1,0)}(0,y) \;=\; \frac{C_F\alpha_s}{\pi}\Big[\Big(\frac{1+y}{y^2}\Big)_+ - \frac{1}{2}\,\delta'(y)\Big]$$

Solution generates an interesting series:

$$\Big[C(x)\Big]_+ \sim \Big[\frac{1}{x^2}\Big]_+ + \Big[\alpha_s \ln(\mu)\frac{\ln x}{x^2} + \ldots\Big]_+ + \Big[\alpha_s^2 \ln^2(\mu)\frac{\ln^2 x}{x^2} + \ldots\Big]_+ + \ldots$$

Summary

- Differential formulation of continuum EFT
  new tools for thinking about field theory modes
- Resolves singularities.
- Interesting applications in B-physics and
  to processes with hard scattering

Open Issues

- Derive factorization theorems for processes with this method
- Level of universality for ψ^{ϵ}
- Factorization theorem with Wilsonian rapidity regulator
- Use of gauge invariant IR regulator, $n^2 \neq 0$, rather than m

THE END