Threshold Resummation in Momentum Space

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TB, M.Neubert, hep-ph/0605050
TB, B. Pecjak and M. Neubert, hep-ph/0607228

Why resummation?

- In problems with widely separated scales $Q_1 >> Q_2$ fixed order PT is not appropriate
  - Large logarithms $\alpha_s^n \, \mathrm{Log}^n(Q_1/Q_2)$ and $\alpha_s^n \, \mathrm{Log}^{2n}(Q_1/Q_2)$. ← Sudakov logarithms
  - Scale in coupling? $\alpha_s(Q_1)$ or $\alpha_s(Q_2)$?
- Standard solution
  - Use effective theories to separate the effects associated with different scales.
  - RG evolution in the effective theory resums large log's.

Resummation for collider processes

- In the past 20 years resummations were performed for many collider processes with scale hierarchies
  - DIS for $x \to 1$, Drell-Yan and Higgs production for $Q^2/s \to 1$, for $Q_T^2/Q^2 \to 0$.
  - e^+e^- event shapes, hadronic event shapes, ...
  - ...
  - LL for arbitrary observable with parton shower
- Resummation are traditionally performed with diagrammatic methods.

Resummation with SCET

- With SCET, we can
  - resum using RG evolution
    - freedom to choose matching scales, simple connection to fixed order
  - properly separate scales
    - no coupling constants at unphysically low scales
  - work directly in momentum space
    - standard approach takes detour into moment space.

Threshold resummation

- Relevant for processes in which

$$Q^2 \gg M_X^2 \gg \Lambda_{\mathrm{QCD}}^2$$

- Factorization theorem takes the form

$$\boxed{d\Gamma = H \cdot J \otimes S}$$

- Will use DIS as an example, but same structure for
  - $B \to X_s\, \gamma$, $B \to X_s\, l^+ l^-$, $B \to X_u\, l\, \nu$,
  - Drell-Yan, Higgs-production, ...

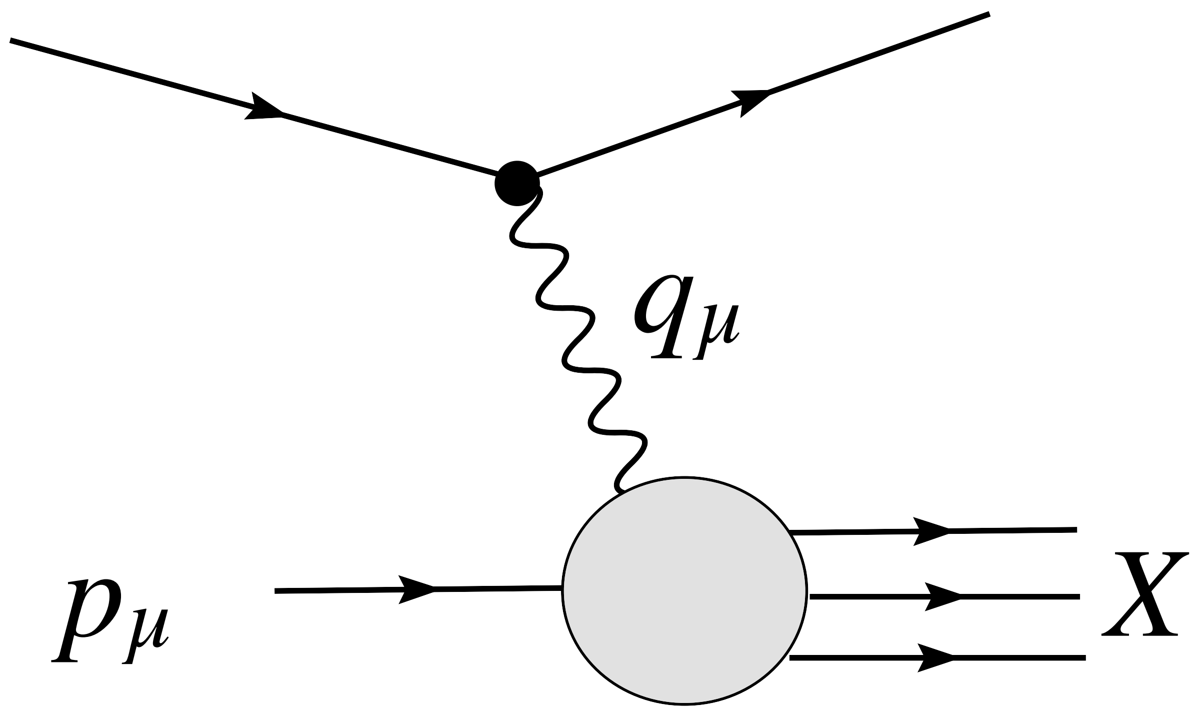
in the limit $M_X^2 << Q^2$.

Outline

- DIS as $x \to 1$
- Resummation
  - Traditional method
  - Using RG evolution in SCET
    - Derivation of RG equations
    - Solution of the RG evolution equations by Laplace transform
  - Connection with trad. method
  - Numerical results

Kinematics of DIS

$$e^{-}(k) + N(p) \to e^{-}(k') + X(P)$$



$$Q^2 = -q^2$$

$$x = \frac{Q^2}{2p \cdot q}$$

- Are interested in the limit $x \to 1$, more precisely $Q^2 \gg Q^2(1-x) \gg \Lambda^2_{QCD}$

$$\approx M_X^2$$

Factorization theorems

- Generic x

$$F_2^{\rm ns} = \int_x^1 \frac{dz}{z}\, \overbrace{C_2\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right)}^{\text{hard scattering coefficient}}\, \overbrace{\frac{x}{z}\phi_q^{\rm ns}(\frac{x}{z}, \mu^2)}^{\text{PDF}}$$

- End-point region $x \to 1$ $\;(Q^2 \gg M_X^2 \gg M_N^2)$

$$F_2^{\rm ns}(x,Q^2) = H(Q^2,\mu)\, Q^2 \int_x^1 \frac{dz}{z}\, J\Big(\underbrace{Q^2\,\frac{1-z}{z}}_{\approx\, M_X^2}, \mu\Big)\, \frac{x}{z}\, \phi_q^{\rm ns}\Big(\frac{x}{z}, \mu\Big)$$

Sterman '87

Traditional method: moment space

Sterman '87, Catani and Trentadue '89

$$F^{\rm ns}_{2,N}(Q^2) = \int_0^1 dx\, x^{N-1} F_2^{\rm ns}(x,Q^2)$$
$$= C_N(Q^2,\mu_f) \sum_q e_q^2\, \phi^{\rm ns}_{q,N}(\mu_f)$$

- Convolution in momentum space $\rightarrow$ product in moment space
- $x \rightarrow 1$ corresponds to $N \rightarrow \infty$. Perturbation theory contains $\alpha_s^n \mathrm{Log}^n(N)$ and $\alpha_s^n \mathrm{Log}^{2n}(N)$
- Split:

$$C_N(Q^2,\mu_f) = g_0(Q^2,\mu_f)\, \exp\left[G_N(Q^2,\mu_f)\right]$$

Resummation in moment space

$$C_N(Q^2,\mu_f) = g_0(Q^2,\mu_f)\,\exp\left[G_N(Q^2,\mu_f)\right]$$

$$G_N(Q^2,\mu_f) = \int_0^1 dz\,\frac{z^{N-1}-1}{1-z}$$
$$\times\left[\int_{\mu_f^2}^{(1-z)Q^2}\frac{dk^2}{k^2}\,A_q(\alpha_s(k)) + B_q\left(\alpha_s(Q\sqrt{1-z})\right)\right]$$

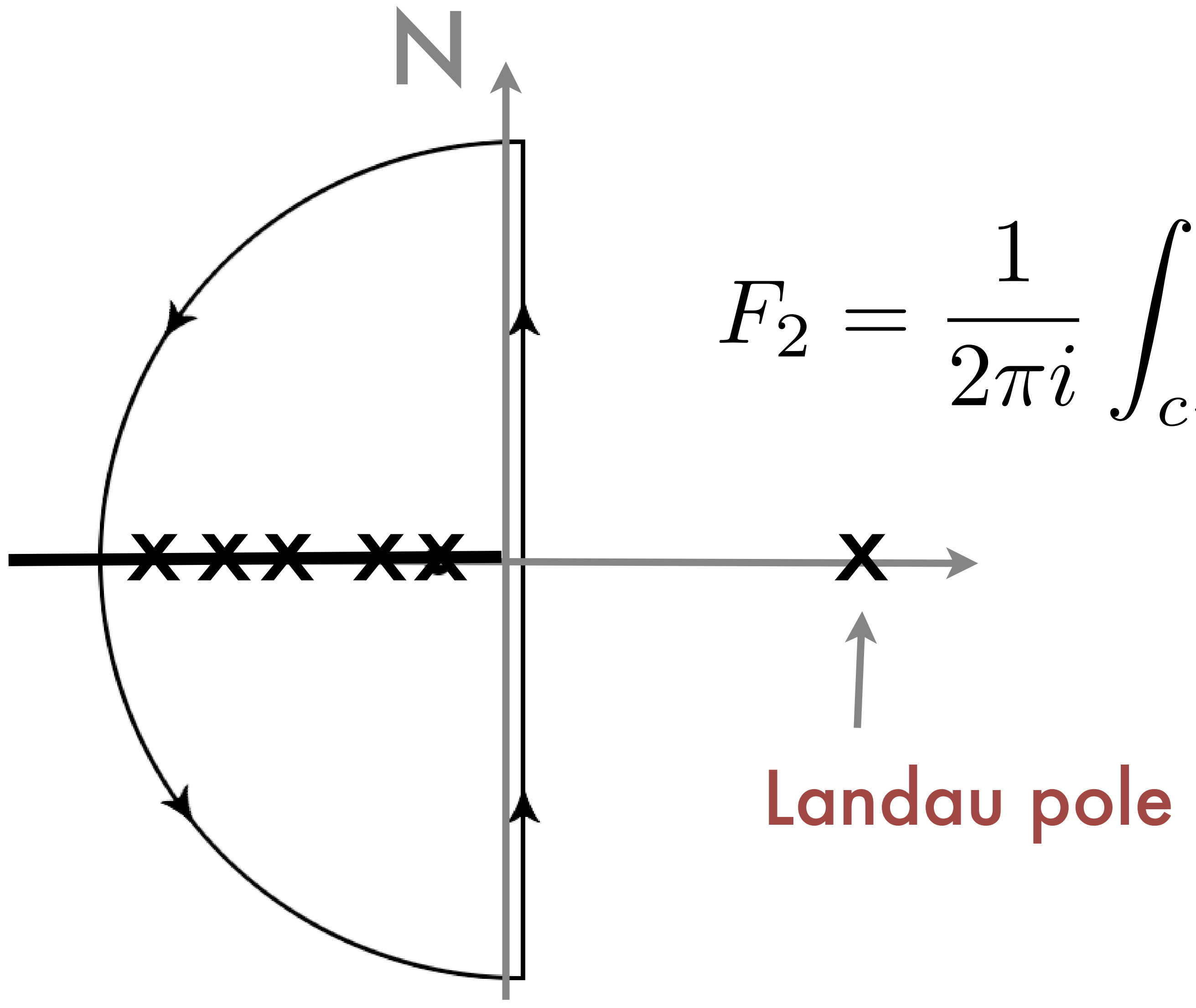
Landau pole

Cusp anomalous dim.

Anom. dim. of ??

- A_q, B_q determined by matching to fixed order result. NNNLL: Moch, Vermaseren, Vogt '05

Mellin Inversion



$$F_2 = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dN\, x^{-N} \phi_N\, C_N$$

- Can only be done numerically
- Problem with Fortran PDF's.

Factorization theorem in SCET

$$F_2^{\rm ns}(x,Q^2) = \sum_q e_q^2\, |C_V(Q^2,\mu)|^2\, Q^2 \int_x^1 d\xi\, J\Big(Q^2\,\frac{\xi-x}{x},\mu\Big)\, \phi_q^{\rm ns}(\xi,\mu)$$

hard part OS form factor	hard-collinear propagator in LC gauge	anti-collinear + soft-collinear PDF for $\xi \to 1$

- Any choice of the scale μ will lead to large perturbative logarithms.
  - Solve RG for individual parts, evolve to common scale.

$$H(\mu_h) \times U_1(\mu_h,\mu_i) \times J(\mu_i) \otimes U_2(\mu_i,\mu_f) \otimes \phi(\mu_f)$$

match $\rightarrow$ **run** $\rightarrow$ **match** $\rightarrow$ **run**

Resummation by RG evolution: 1. hard part

- RG equation for C_V

$$\frac{d}{d\ln\mu}C_V(Q^2,\mu)=\left[\Gamma_{\text{cusp}}(\alpha_s)\,\ln\frac{Q^2}{\mu^2}+\gamma^V(\alpha_s)\right]C_V(Q^2,\mu)$$

- Solution

$$C_V(Q^2,\mu)=\exp\left[2S(\mu_h,\mu)-a_{\gamma^V}(\mu_h,\mu)\right]\left(\frac{Q^2}{\mu_h^2}\right)^{-a_\Gamma(\mu_h,\mu)}C_V(Q^2,\mu_h)$$

$$S(\nu,\mu)=-\int\limits_{\alpha_s(\nu)}^{\alpha_s(\mu)}d\alpha\,\frac{\Gamma_{\text{cusp}}(\alpha)}{\beta(\alpha)}\int\limits_{\alpha_s(\nu)}^{\alpha}\frac{d\alpha'}{\beta(\alpha')}\,,\qquad a_\Gamma(\nu,\mu)=-\int\limits_{\alpha_s(\nu)}^{\alpha_s(\mu)}d\alpha\,\frac{\Gamma_{\text{cusp}}(\alpha)}{\beta(\alpha)}$$

Three-loop anomalous dimension

- On-shell form factor is known to two-loops, divergencies even to three loops (Moch, Vermaseren Vogt '05).
- Can extract anomalous dimension to three loops:

$$
\begin{aligned}
\gamma_0^V &= -6C_F = -8\,,\\
\gamma_1^V &= C_F^2\left(-3+4\pi^2-48\zeta_3\right)+C_FC_A\left(-\frac{961}{27}-\frac{11\pi^2}{3}+52\zeta_3\right)+C_FT_Fn_f\left(\frac{260}{27}+\frac{4\pi^2}{3}\right)\\
&\approx 1.1419\,,\\
\gamma_2^V &= C_F^3\left(-29-6\pi^2-\frac{16\pi^4}{5}-136\zeta_3+\frac{32\pi^2}{3}\zeta_3+480\zeta_5\right)\\
&\quad+C_F^2C_A\left(-\frac{151}{2}+\frac{410\pi^2}{9}+\frac{494\pi^4}{135}-\frac{1688}{3}\zeta_3-\frac{16\pi^2}{3}\zeta_3-240\zeta_5\right)\\
&\quad+C_FC_A^2\left(-\frac{139345}{1458}-\frac{7163\pi^2}{243}-\frac{83\pi^4}{45}+\frac{7052}{9}\zeta_3-\frac{88\pi^2}{9}\zeta_3-272\zeta_5\right)\\
&\quad+C_F^2T_Fn_f\left(\frac{5906}{27}-\frac{52\pi^2}{9}-\frac{56\pi^4}{27}+\frac{1024}{9}\zeta_3\right)\\
&\quad+C_FC_AT_Fn_f\left(-\frac{34636}{729}+\frac{5188\pi^2}{243}+\frac{44\pi^4}{45}-\frac{3856}{27}\zeta_3\right)\\
&\quad+C_FT_F^2n_f^2\left(\frac{19336}{729}-\frac{80\pi^2}{27}-\frac{64}{27}\zeta_3\right)\approx -249.388\,.
\end{aligned}
$$

Aside: derivation of the RG evolution equation

- Off-shell vector form factor

$$F(Q^2, p^2, p'^2, \mu) = C_V(Q^2, \mu)\, \hat{J}(p^2, \mu)\, \hat{J}(p'^2, \mu)\, S(p_{\rm soft}^2, \mu)$$

not the same collinear matrix element as in DIS

$$p_{\rm soft}^2 = \frac{p^2 p'^2}{Q^2}$$

- Soft matrix element is Wilson line with a cusp. RG equation:

$$\frac{d}{d\ln\mu} S(p_{\rm soft}^2, \mu) = -\left[\Gamma_{\rm cusp} \ln\frac{\mu^2}{p_{\rm soft}^2} + \gamma_s\right] S(p_{\rm soft}^2, \mu)$$

Korchemsky & Radyushkin '87

Anomalous dimensions for J and C_V

TB, Hill, Lange, Neubert '03

$$\frac{d}{d\ln\mu}\ln\Big[C_V(Q^2,\mu)\,\hat{J}(p^2,\mu)\,\hat{J}(p'^2,\mu)\,S(p^2_{\rm soft},\mu)\Big] =$$
$$\gamma_V(Q^2,\mu)+\gamma_{\hat{J}}(p^2,\mu)+\gamma_{\hat{J}}(p'^2,\mu)-\Gamma_{\rm cusp}\ln\frac{Q^2\mu^2}{p^2p'^2}-\gamma_s=0$$

- Anomalous dimensions

$$\gamma_V(Q^2,\mu)=-\Gamma_{\rm cusp}\ln\frac{\mu^2}{Q^2}+\gamma_V$$

$$\gamma_{\hat{J}}(p^2,\mu)=\Gamma_{\rm cusp}\ln\frac{\mu^2}{p^2}+\gamma_{\hat{J}}$$

Resummation by RG evolution: 2. jet function

- RG evolution equation for jet-function involves convolution

$$\frac{d}{d\ln\mu} J(p^2,\mu) \equiv \gamma \otimes J$$

$$= \int_0^{p^2} dp'^2\, \gamma(p^2 - p'^2, \mu)\, J(p'^2, \mu)$$

$$\gamma(p^2 - p'^2, \mu) = 2\Gamma_{\rm cusp}(\alpha_s) \left(\frac{1}{p^2 - p'^2}\right)_*^{[\mu^2]} + 2\gamma^J(\alpha_s)\,\delta(p^2 - p'^2)$$

similar to plus distribution

Convoluted RG-evolution

- RG evolution equations for
  - jet-function(s)
  - shape function
  - pdf in the end-point
  - *B*-meson LCDA

all have the same structure.

- Will now discuss solution in detail
  - First solved by Lange & Neubert '03. Have simpler derivation based on Laplace transform.

TB, M.Neubert '06

Engineering 101: Laplace transformation

- Definition

$$\mathcal{L}[f](s) = \int_0^\infty d\omega\, e^{-s\omega}\, f(\omega)$$

- Inversion

$$f(\omega) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds\, e^{sx}\, \mathcal{L}[f](s)$$

- De-convolution

$$\mathcal{L}[f \otimes g](s) = \mathcal{L}[f](s) \times \mathcal{L}[g](s)$$

$$f \otimes g = \int_0^\omega d\omega'\, f(\omega - \omega')\, g(\omega')$$

Star distributions

$$\int_0^\Omega d\omega \left(\frac{1}{\omega}\right)_*^{[\mu]} f(\omega) = \int_0^\Omega d\omega \frac{1}{\omega}\left[f(\omega) - f(0)\right] + \ln\frac{\Omega}{\mu} f(0)$$

- for Ω=μ=1 the *-dist's reduce to +-dist's
- Generating function

$$\int_0^\Omega d\omega \frac{\mu^\epsilon}{\omega^{1+\epsilon}} f(\omega) = \int_0^\Omega d\omega \left[-\frac{1}{\epsilon}\delta(\omega) + \left(\frac{1}{\omega}\right)_*^{[\mu]} + \sum_{n=1}^{\infty}\frac{(-\epsilon)^n}{n!}\left(\frac{\ln^n(\omega)}{\omega}\right)_*^{[\mu]}\right] f(\omega)$$

- Laplace transforms:

$$\begin{aligned}
\mathcal{L}[\delta(\omega)] &= 1 \\
\mathcal{L}\left[\left(\frac{1}{\omega}\right)_*^{[\mu]}\right] &= -\gamma_E - \ln(\mu s) \\
\mathcal{L}\left[\left(\frac{\ln\omega}{\omega}\right)_*^{[\mu]}\right] &= \frac{\pi^2}{6} + \ln^2(e^{\gamma_E}\mu s)
\end{aligned}$$

Application to jet-function RG

- Laplace

$$\tilde{j}\Big(\ln\frac{Q^2}{\mu^2},\mu\Big) = \int_0^\infty dp^2\, e^{-sp^2}\, J(p^2,\mu)\,, \qquad s = \frac{1}{e^{\gamma_E} Q^2}$$

- Same RGE as hard function

$$\frac{d}{d\ln\mu}\tilde{j}\Big(\ln\frac{Q^2}{\mu^2},\mu\Big) = -\left[2\Gamma_{\rm cusp}(\alpha_s)\,\ln\frac{Q^2}{\mu^2} + 2\gamma^J(\alpha_s)\right]\tilde{j}\Big(\ln\frac{Q^2}{\mu^2},\mu\Big)$$

$$\tilde{j}\Big(\ln\frac{Q^2}{\mu^2},\mu\Big) = \exp\left[-4S(\mu_i,\mu) + 2a_{\gamma^J}(\mu_i,\mu)\right]\left(\frac{Q^2}{\mu_i^2}\right)^{2a_\Gamma(\mu_i,\mu)}\tilde{j}\Big(\ln\frac{Q^2}{\mu_i^2},\mu_i\Big)$$

- Invert: $\eta = 2a_\Gamma(\mu_i,\mu)$.

$$J(p^2,\mu) = \exp\left[-4S(\mu_i,\mu) + 2a_{\gamma^J}(\mu_i,\mu)\right]\frac{e^{-\gamma_E\eta}}{\Gamma(\eta)}\int_0^{p^2} dp'^2\,\frac{J(p'^2,\mu_i)}{(\mu_i^2)^\eta (p^2-p'^2)^{1-\eta}}$$

Even more elegantly:

- Rewrite log's as derivatives:

$$\tilde{j}(\ln \frac{Q^2}{\mu_i^2}, \mu_i) \left(\frac{Q^2}{\mu_i}\right)^\eta = \tilde{j}(\partial_\eta, \mu_i) \left(\frac{Q^2}{\mu_i}\right)^\eta \qquad \eta = 2a_\Gamma(\mu_i, \mu).$$

- Solution

$$J(p^2, \mu) = \exp\left[-4S(\mu_i, \mu) + 2a_{\gamma^J}(\mu_i, \mu)\right] \times \widetilde{j}(\partial_\eta, \mu_i) \frac{e^{-\gamma_E \eta}}{\Gamma(\eta)} \frac{1}{p^2} \left(\frac{p^2}{\mu_i^2}\right)^\eta ,$$

Resummation by RG evolution: 3. PDF near the end-point

$$\frac{d}{d\ln\mu}\,\phi_q^{\rm ns}(\xi,\mu) \;=\; \int_\xi^1 \frac{dz}{z}\,P_{q\leftarrow q}^{(\rm endpt)}(z)\,\phi_q^{\rm ns}\Big(\frac{\xi}{z},\mu\Big)$$

$$P_{q\leftarrow q}^{(\rm endpt)}(z) = \frac{2\Gamma_{\rm cusp}(\alpha_s)}{(1-z)_+} + 2\gamma^\phi(\alpha_s)\,\delta(1-z)$$

- Equation (and its solution) can be obtained from

$$\frac{d}{d\ln\mu}F_2(x,Q^2) = 0$$

- Can obtain 3-loop γ^J using

$$\gamma^J = \gamma^\phi + \gamma^V$$

Moch, Vermaseren Vogt '04

Result for F_2

- Evolve C_V and J from μ_h and μ_i to scale μ_f, plug into factorization theorem

$$F_2^{\rm ns}(x,Q^2) = \sum_q e_q^2\,|C_V(Q^2,\mu_h)|^2\,U(Q,\mu_h,\mu_i,\mu_f)$$
$$\times\widetilde{j}\Big(\ln\frac{Q^2}{\mu_i^2}+\partial_\eta,\mu_i\Big)\,\frac{e^{-\gamma_E\eta}}{\Gamma(\eta)}\int_x^1 d\xi\,\frac{\phi_q^{\rm ns}(\xi,\mu_f)}{(\xi-x)^{1-\eta}}$$

$$U(Q,\mu_h,\mu_i,\mu_f) = \exp\left[4S(\mu_h,\mu_i)-2a_{\gamma_V}(\mu_h,\mu_i)\right]$$
$$\times\left(\frac{Q^2}{\mu_h^2}\right)^{-2a_\Gamma(\mu_h,\mu_i)}\exp\left[2a_{\gamma^\phi}(\mu_i,\mu_f)\right]\,,$$

Result

- If we assume $\phi_q(x,\mu_f) \sim (1-x)^{b(\mu_f)}$:

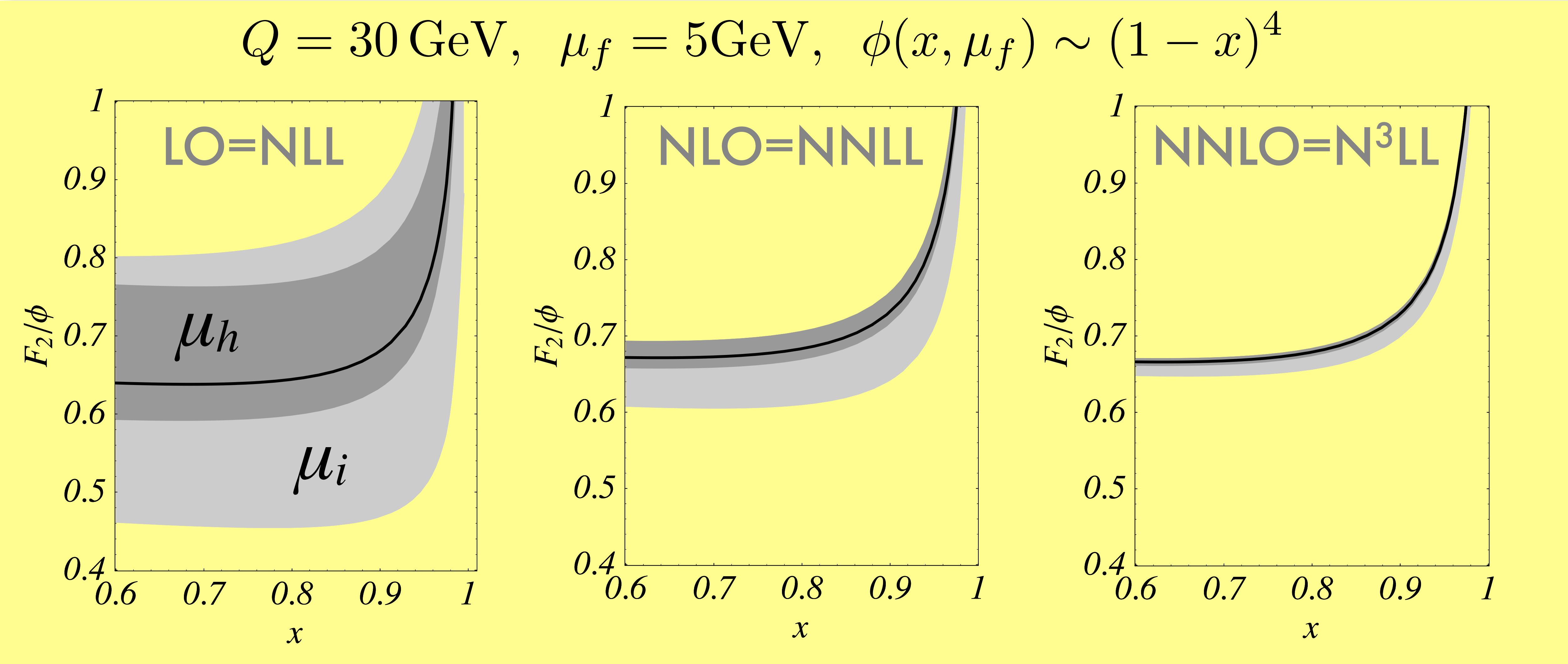
$$\frac{F_2^{\rm ns}(x,Q^2)}{\sum_q e_q^2\, x\, \phi_q^{\rm ns}(x,\mu_f)} = |C_V(Q^2,\mu_h)|^2\, U(Q,\mu_h,\mu_i,\mu_f)$$
$$\times (1-x)^\eta\, \widetilde{j}\Big(\ln\frac{Q^2(1-x)}{\mu_i^2}+\partial_\eta,\mu_i\Big)$$
$$\times \frac{e^{-\gamma_E\eta}\,\Gamma(1+b(\mu_f))}{\Gamma(1+b(\mu_f)+\eta)}\,.$$

- Resummed result obtained after plugging in fixed order results for coefficient C_V, jet-function and anom. dimensions.

Difference to traditional approach

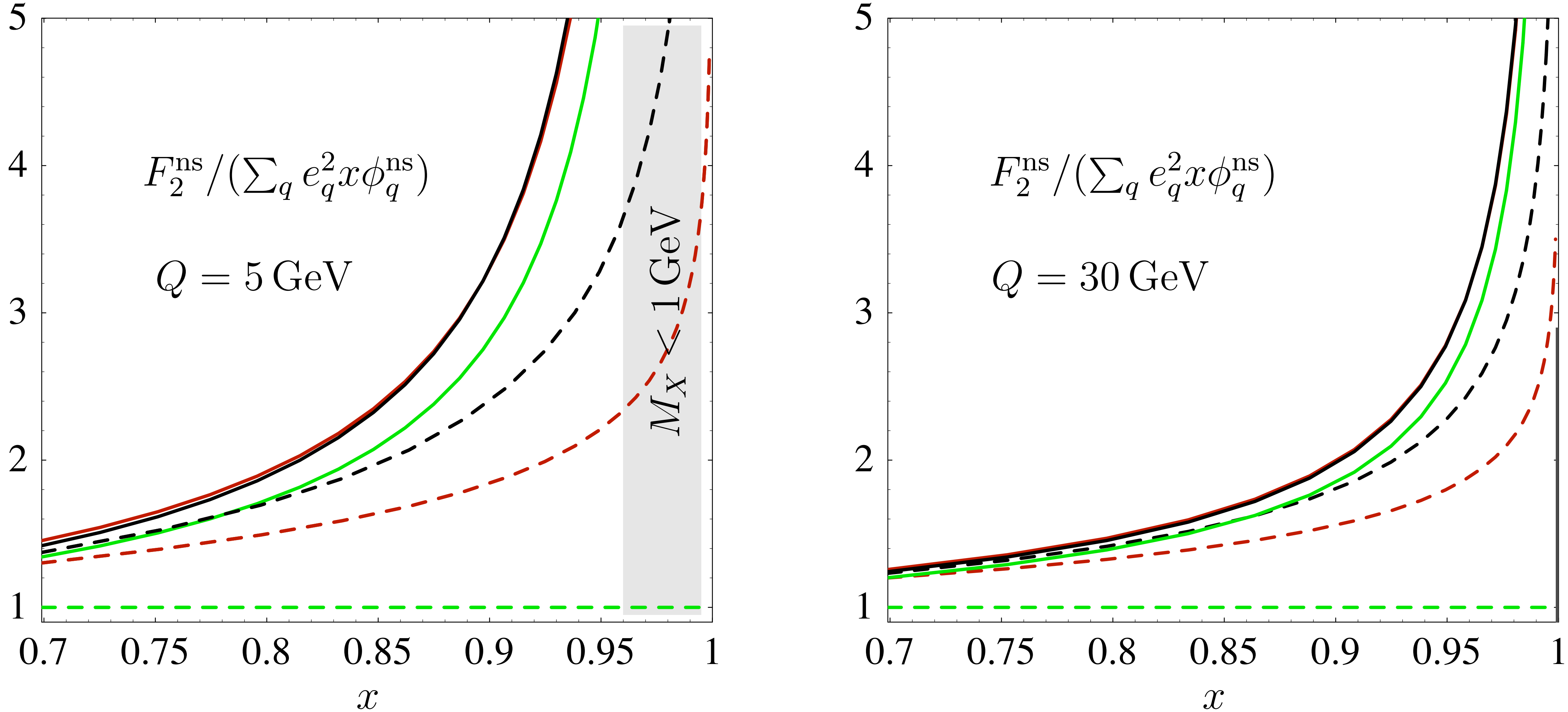
- Simple analytic result in momentum space
- No Landau pole ambiguities. No coupling constant below scales μ_h, μ_i and μ_f.
- Freedom to choose scales μ_h, μ_i and μ_f
  - Obtain fixed order for $\mu_h=\mu_i=\mu_f$. Trivial matching to fixed order result for generic x.
  - Set appropriate scales *after* integrating
    - Avoids large spurious power corrections discussed by Catani et al. hep-ph/9604351
  - Estimate uncertainties with scale variation

Result for $F_2^{\mathrm{ns}}(\mathrm{x})/\phi_q(x)$



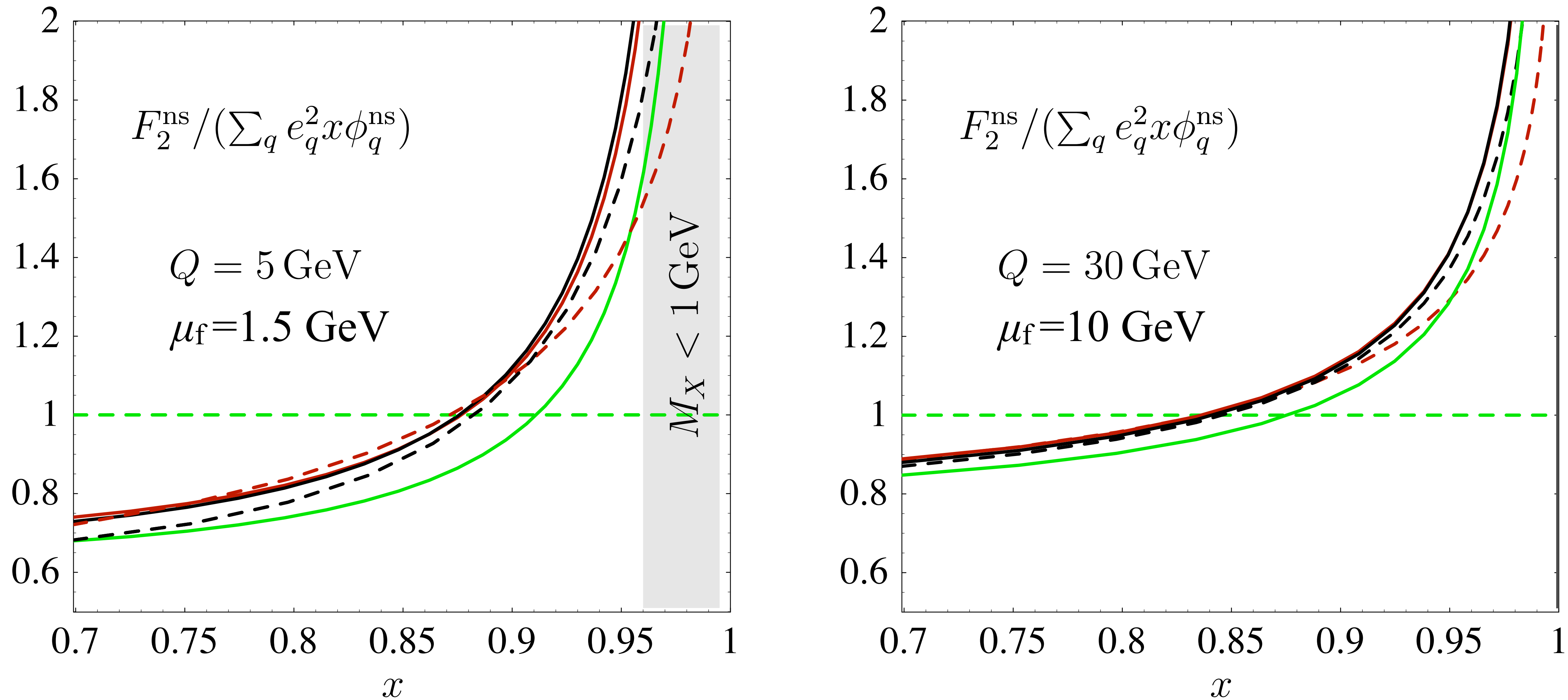
- Default scales: $\mu_h^2=Q^2$ and $\mu_i^2=Q^2(1\text{-x})$
  - Bands obtained by varying these scales a factor of two up and down.
  - Matching scales are fixed in traditional approach.

Comparison with fixed order, μ_f=Q



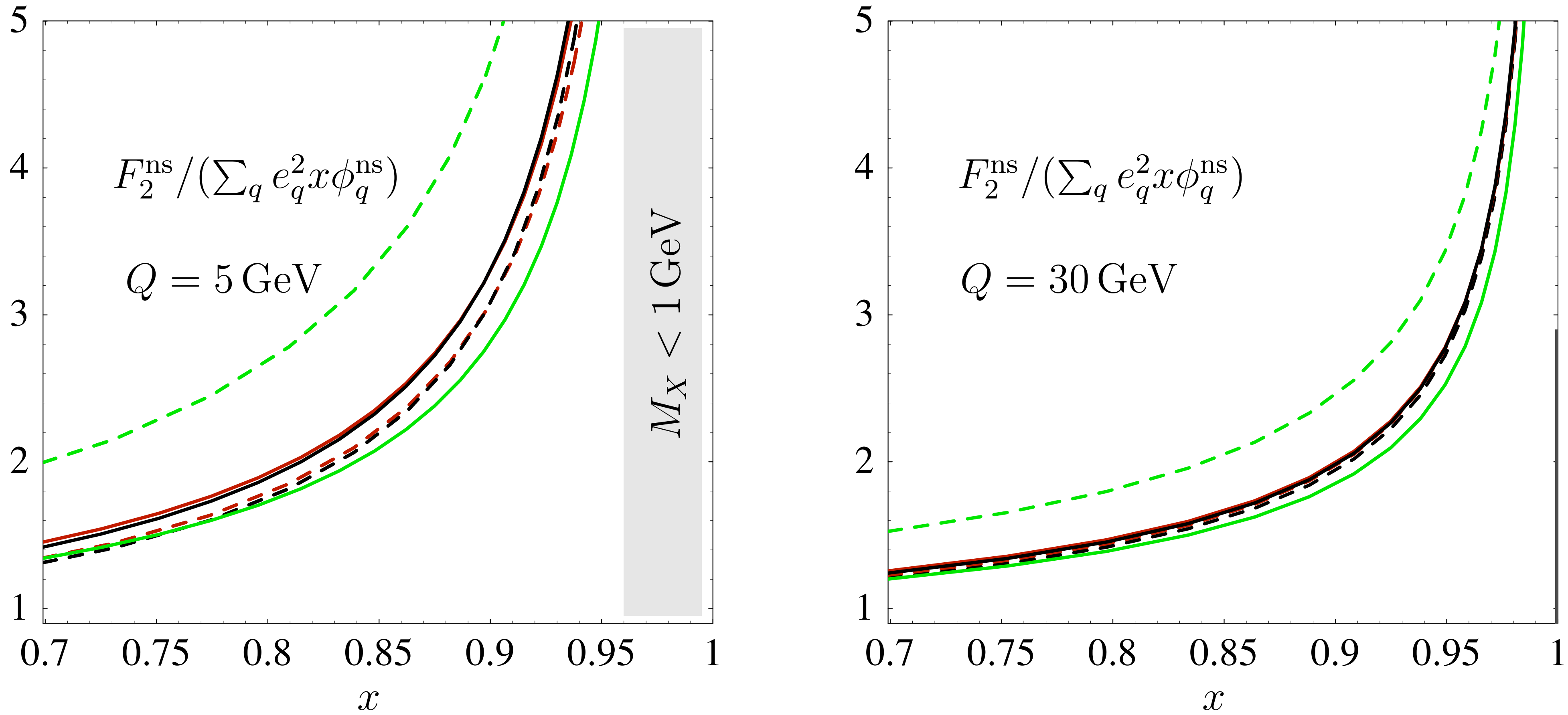
- LO (=NLL), NLO, NNLO
- Dashed: fixed order. Solid: resummed.
- Large K-factors.

Comparison with fixed order, low μ_f



- LO (=NLL), NLO, NNLO
- Dashed: fixed order. Solid: resummed.
- Fixed order with $\mu=\mu_{\rm f}$ fairly close to resummed result!

Comparison with moment space result



- Dashed: Mellin inverted moment space results. Solid: momentum space results.
- Only small numerical differences (different scale choice, 1/N corrections in moment space).
- Faster convergence of momentum space results.

Connection with standard approach

- Can derive traditional expression for resummation in moment space from SCET. With $\mu_h = Q^2$ $\mu_i = Q^2/N$

$$G_N^{\mathrm{SCET}}(Q^2,\mu_f) = \int_{Q^2/\bar{N}}^{Q^2} \frac{dk^2}{k^2}\left[\ln\frac{k^2}{Q^2}\,\Gamma_{\mathrm{cusp}}(\alpha_s(k)) - \gamma^J(\alpha_s(k)) - \frac{d\ln\widetilde{j}(0,k)}{d\ln k^2}\right] - \ln\bar{N}\int_{\mu_f^2}^{Q^2/\bar{N}}\frac{dk^2}{k^2}\,\Gamma_{\mathrm{cusp}}(\alpha_s(k))\,,$$

- Note different form of exponent

$$G_N(Q^2,\mu_f) = \int_0^1 dz\,\frac{z^{N-1}-1}{1-z}\left[\int_{\mu_f^2}^{(1-z)Q^2}\frac{dk^2}{k^2}\,A_q(\alpha_s(k)) + B_q\left(\alpha_s(Q\sqrt{1-z})\right)\right]$$

Connection with standard approach

- Can relate EFT expression to standard result. The two agree provided that

$$\left(1+\frac{\pi^2}{12}\nabla^2+\dots\right)B_q(\alpha_s)=\gamma^J(\alpha_s)+\nabla\,\ln\widetilde{j}(0,\mu)$$
$$-\left(\frac{\pi^2}{12}\nabla-\frac{\zeta_3}{3}\nabla^2+\dots\right)\Gamma_{\rm cusp}(\alpha_s)\,,\qquad \nabla=d/d\ln\mu^2.$$

- fulfilled with two-result from explicit calculation of $J(p^2)$.
- coefficient B_q is not an anomalous dimension

Momentum space?

- Past controversy about performing resummations in momentum space. Claims that

  1. exponentiation is incomplete
  2. momentum conservation is violated
  3. there are large ambiguities, not related to Landau pole singularities.

  Catani, Mangano, Nason, Trentadue '96

- 3. are not present in our formalism. Not sure what 1. and 2. mean.

Integral over structure function at LL

$$\mathcal{F}_2^{\rm ns}(x,Q^2) = \int_{1-x}^1 dy\, F_2^{\rm ns}(y,Q^2)$$

- LL, expand exponent in $a = \Gamma_0 \frac{\alpha_s(Q)}{8\pi}$

$$\mathcal{F}_2^{\rm ns}(x,Q^2) = \int_{1-x}^1 dy \sum_q e_q^2\, y\, \phi_q^{\rm ns}(y,Q)\, \exp\left[-a \ln^2 \frac{\mu_i^2}{\mu_h^2} + 2a \ln \frac{\mu_i^2}{\mu_f^2}\, \ln(1-y)\right]$$

- With scale choice $\mu_f = \mu_h = Q,\ \mu_i \approx Q\sqrt{1-y}$

$$= \int_{1-x}^1 dy \sum_q e_q^2\, y\, \phi_q^{\rm ns}(y,\mu_f)\, \exp\left[a \ln^2(1-y)\right]$$

Nonintegrable singularity!

- Choose scales after integration!

Summary

- Traditionally, resummation for hard processes is performed in moment space.
  - Landau poles (in Sudakov exponent and Mellin inversion)
  - Mellin inversion only numerically
- Solving RG equations in SCET, we have obtained resummed expressions directly in momentum space.
  - Clear scale separation. No Landau pole ambiguities.
  - Simple analytic expressions.
  - Trivial connection with fixed order expressions.
- Same technology is applicable to many other processes.
  - See Matthias's talk